GateXAIML
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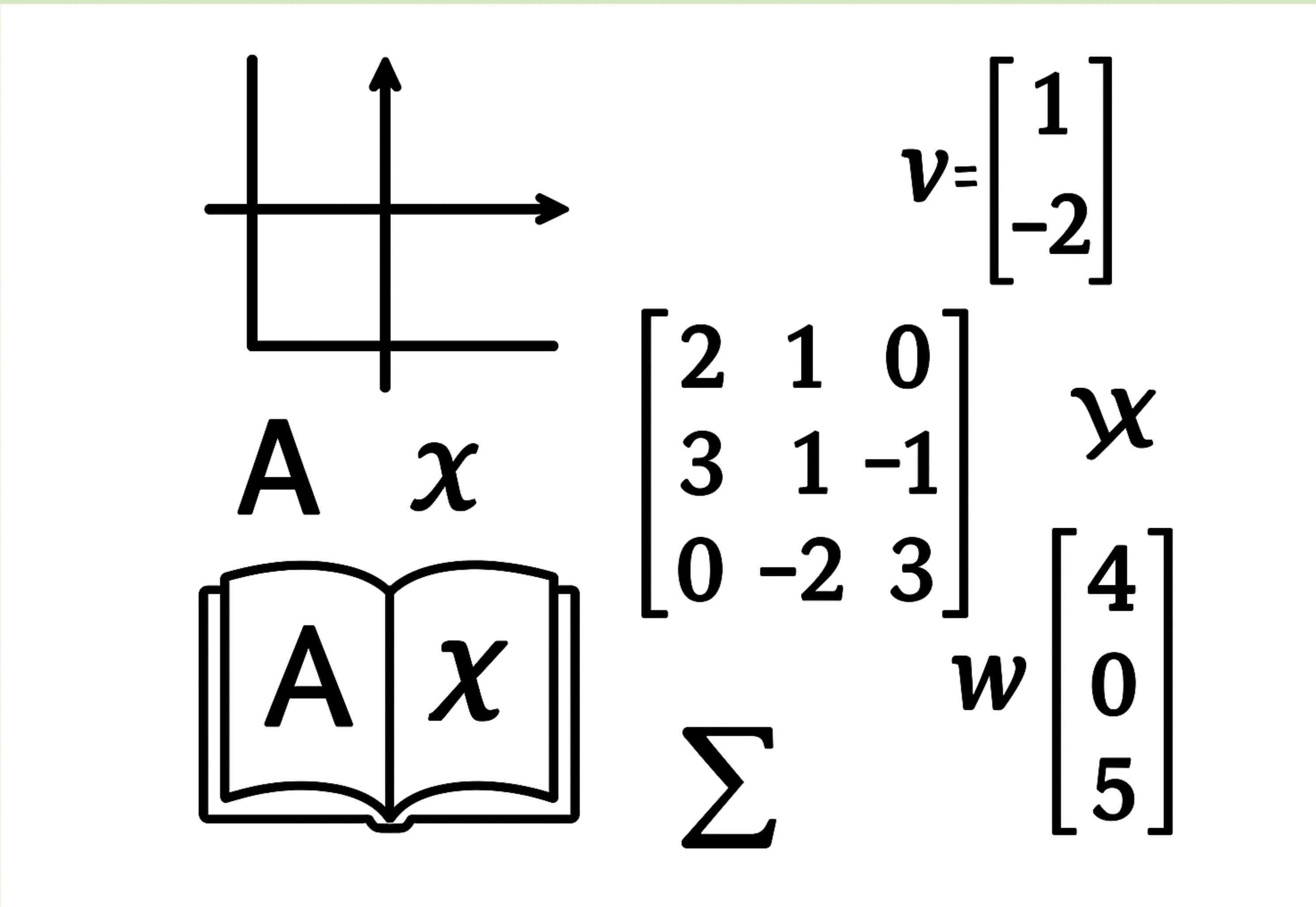
GATE – Computer Science Engineering (CSE)

Matrices and Linear Algebra



GateXAIML

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Contents

About the Book

The **GATE Computer Science and Engineering (CSE) exam** serves as a national-level gateway to higher studies, research, and employment in top institutions and organizations. It evaluates a candidate's understanding of core Computer Science subjects, including Calculus, Matrices and Linear Algebra, Probability and Statistics, Discrete Mathematics, Algorithms, Data Structures, Theory of Computation, Databases, Operating Systems, and Computer Networks.

This book is **designed for aspirants of the GATE CSE exam**, focusing on **Matrices and Linear Algebra**. It systematically covers theory, solved examples, and practice problems **aligned with the official syllabus**, helping learners build strong conceptual foundations and problem-solving skills.

Selected solutions and topic-wise lectures will be explained on my YouTube channel (@GATEXAiml), providing a **complete resource for GATE CSE preparation**.

Dedicated to all my Gurus and Students.
"Knowledge grows only when shared — and it must remain free, for that is how it thrives."

Linear Algebra (Matrices and Linear Algebra) - Syllabus

Matrices, determinants, system of linear equations, eigenvalues and eigenvectors, LUdecomposition. properties.

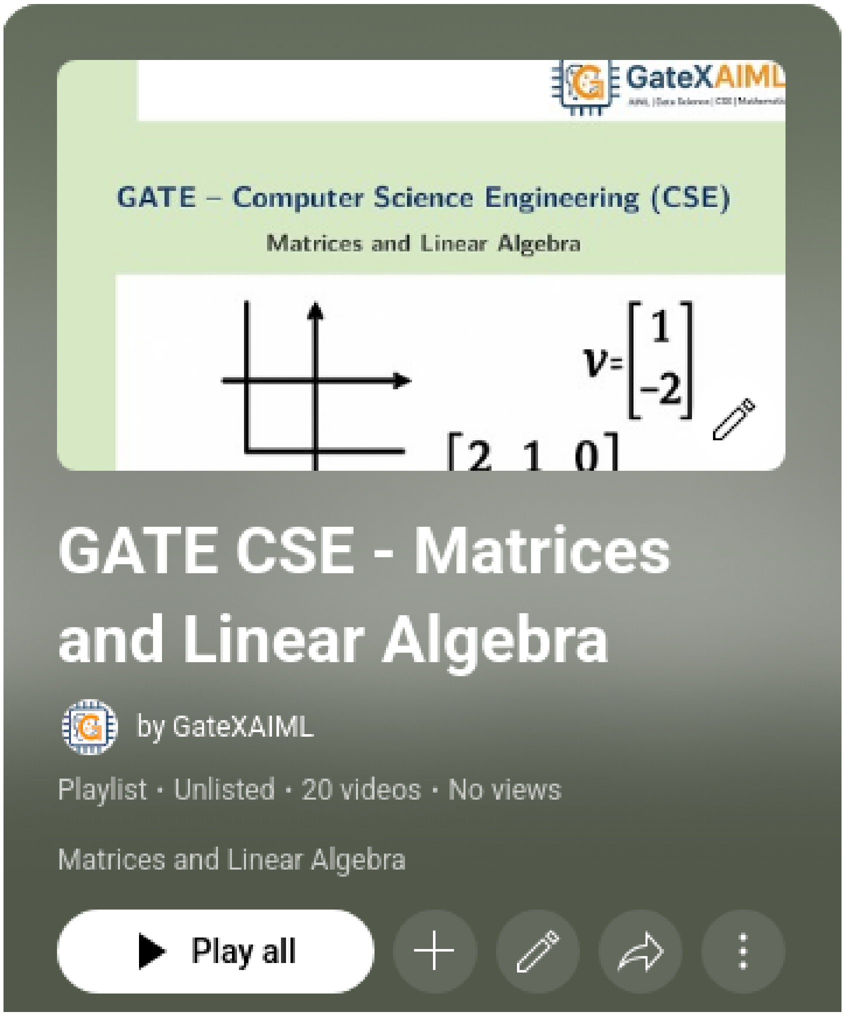
STOP!

Attention!

Some examples solved in video lectures are different from those given in this book.

The procedure to solve problems and examples is well explained in the video lectures, and it is highly recommended to go through the video lectures for complete understanding.

Official Video Playlist



Watch on YouTube

Chapter 1

Matrices and Determinants

1.1 Introduction

A **matrix** is a rectangular arrangement of numbers in rows and columns. It is usually enclosed within square or round brackets. If a matrix has m rows and n columns, it is called an $m \times n$ ("m by n") matrix.

Concept

An $m \times n$ matrix A is written as:

$$A = [a_{ij}]_{m\times n}, \quad \text{where } a_{ij} \text{ is the element in the } i^{th} \text{ row and } j^{th} \text{ column.}$$

Example 1:

A 2×3 matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

1.2 Types of Matrices

Row Matrix

A matrix with only one row is called a **row matrix**.

Concept

If A is a $1 \times n$ matrix, then A is a row matrix.

Example 2:

$$A = \begin{bmatrix} 2 & 5 & -3 & 7 \end{bmatrix}$$

is a row matrix of order 1×4.

Column Matrix

A matrix with only one column is called a **column matrix**.

Concept

If A is an $m \times 1$ matrix, then A is a column matrix.

Example 3:

$$B = \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}$$

is a column matrix of order 3×1.

Square Matrix

A matrix with the same number of rows and columns is called a **square matrix**.

Concept

An $n \times n$ matrix is called a square matrix of order n.

Example 4:

$$C = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$$

is a square matrix of order 3.

Zero Matrix

A matrix in which all elements are zero is called a **zero matrix** or **null matrix**.

Concept

If $O = [0]_{m\times n}$, then O is a zero matrix of order $m \times n$.

Example 5:

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a zero matrix of order 2×3.

Identity Matrix

A square matrix in which all diagonal elements are 1 and all other elements are 0 is called an **identity matrix**.

Concept

The identity matrix of order n is denoted by I_n, where

$$(I_n)_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Example 6:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the 3×3 identity matrix.

Diagonal Matrix

A square matrix in which all non-diagonal elements are zero is called a **diagonal matrix**.

Concept

If $A = [a_{ij}]_{n\times n}$ and $a_{ij} = 0$ for all $i \neq j$, then A is a diagonal matrix.

Example 7:

$$E = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

is a diagonal matrix of order 3.

Triangular Matrix

A **triangular matrix** is a square matrix in which all the entries either below or above the main diagonal are zero.

Upper Triangular Matrix

A square matrix $A = [a_{ij}]_{n\times n}$ is called an **upper triangular matrix** if $a_{ij} = 0$ for all $i > j$ (all elements below the main diagonal are zero).

Example 8:

An upper triangular matrix of order 3:

$$U = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 7 \end{bmatrix}$$

Here, all elements below the main diagonal are 0.

Lower Triangular Matrix

A square matrix $A = [a_{ij}]_{n\times n}$ is called a **lower triangular matrix** if $a_{ij} = 0$ for all $i < j$ (all elements above the main diagonal are zero).

Example 9:

A lower triangular matrix of order 3:

$$L = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 5 & 0 \\ 7 & 1 & 6 \end{bmatrix}$$

Here, all elements above the main diagonal are 0.

Concept

Every diagonal matrix is both an upper triangular and a lower triangular matrix. That is, diagonal matrices are a special case of triangular matrices.

1.3 Matrix Operations

Addition

Two matrices of the same order can be added by adding their corresponding elements.

Concept

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, then

$$A + B = [a_{ij} + b_{ij}]_{m\times n}.$$

Example 10:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Scalar Multiplications

Multiplying a matrix by a scalar means multiplying each entry of the matrix by that scalar.

Concept

If k is a scalar and $A = [a_{ij}]_{m\times n}$, then

$$kA = [k \cdot a_{ij}]_{m\times n}.$$

Example 11:

$$3 \cdot \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 0 & 12 \end{bmatrix}$$

Matrix Multiplication

Matrix multiplication is defined only when the number of columns of the first matrix equals the number of rows of the second.

Concept

If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then AB is an $m \times p$ matrix given by:

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Example 12:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

1.4 Transpose

The transpose of a matrix is obtained by interchanging its rows and columns.

Concept

If $A = [a_{ij}]_{m \times n}$, then its transpose is

$$A^T = [a_{ji}]_{n \times m}.$$

Example 13:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Concept

Let A and B be matrices and let k be a scalar. Then, whenever the sum and product are defined, the following properties hold:

1. $(A + B)^T = A^T + B^T$
2. $(A^T)^T = A$
3. $(kA)^T = kA^T$
4. $(AB)^T = B^T A^T$

Note: In (4), the transpose of a product is the product of the transposes, but in the *reverse order*.

1.5 Square Matrices

A **square matrix** is a matrix with the same number of rows and columns. An $n \times n$ square matrix is said to be of **order** n and is sometimes called an n-**square matrix**.

Concept

For square matrices of order n, the operations of addition, multiplication, scalar multiplication, and transpose are always defined, and the result is again an $n \times n$ matrix.

Example 14:

The following are square matrices of order 3:

$$A = \begin{bmatrix} 1 & 2 & -4 \\ 5 & 6 & 3 \\ 3 & -4 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -5 & 0 \\ 3 & 1 & 2 \\ 3 & 1 & -4 \end{bmatrix}$$

Examples of operations:

$$A^T = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 6 & -4 \\ -4 & 3 & 7 \end{bmatrix}, \quad 2A = \begin{bmatrix} 2 & 4 & -8 \\ 10 & 12 & 6 \\ 6 & -8 & 14 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 3 & -3 & -4 \\ 8 & 7 & 5 \\ 6 & -3 & 3 \end{bmatrix}, \quad AB = \begin{bmatrix} -12 & 0 & 20 \\ -27 & -30 & -33 \\ 17 & 7 & -35 \end{bmatrix}$$

$$BA = \begin{bmatrix} -22 & -24 & -26 \\ 10 & 12 & 14 \\ 27 & 30 & 33 \end{bmatrix}$$

1.6 Diagonal and Trace

The **diagonal** or **main diagonal** of an $n \times n$ matrix $A = [a_{ij}]$ consists of the elements $a_{11}, a_{22}, \ldots, a_{nn}$.

The **trace** of A, written $\mathrm{tr}(A)$, is the sum of the diagonal elements:

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

Concept

Suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ are $n \times n$ matrices, and k is a scalar. Then:

$$\begin{aligned} &\text{(i)} && \mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B), \\ &\text{(ii)} && \mathrm{tr}(kA) = k\ \mathrm{tr}(A), \\ &\text{(iii)} && \mathrm{tr}(A^T) = \mathrm{tr}(A), \\ &\text{(iv)} && \mathrm{tr}(AB) = \mathrm{tr}(BA). \end{aligned}$$

Example 15:

For matrices A and B

$$A = \begin{bmatrix} 1 & 2 & -4 \\ 5 & 6 & 3 \\ 3 & -4 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -5 & 0 \\ 3 & 1 & 2 \\ 3 & 1 & -4 \end{bmatrix}$$

Diagonal of A: $\{1, -4, 7\}$, so $\mathrm{tr}(A) = 1 + (-4) + 7 = 4$.
Diagonal of B: $\{2, 1, -4\}$, so $\mathrm{tr}(B) = 2 + 1 - 4 = -1$.
Now:

$$\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B) = 4 + (-1) = 3$$

$$\mathrm{tr}(2A) = 2 \cdot \mathrm{tr}(A) = 8$$

$$\mathrm{tr}(A^T) = \mathrm{tr}(A) = 4$$

$$\mathrm{tr}(AB) = \mathrm{tr}(BA) = -30$$

1.7 Identity and Scalar Matrices

The $n \times n$ **identity matrix**, denoted I_n, has 1's on the diagonal and 0's elsewhere:

$$(I_n)_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Concept

For any $n \times n$ matrix A,

$$AI_n = I_nA = A$$

More generally, for any scalar k, the matrix kI with k's on the diagonal and 0's elsewhere is called a **scalar matrix**.

Concept

Multiplying a matrix A by kI is equivalent to multiplying A by k:

$$(kI)A = k(IA) = kA$$

Example 16:

Identity and scalar matrices:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad 5I_3 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad 5I_4 = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

1.8 Powers of Matrices

Let A be an $n \times n$ matrix over a field K. The powers of A are defined as:

$$A^2 = A \cdot A, \quad A^3 = A^2 \cdot A, \quad \ldots, \quad A^{n+1} = A^n \cdot A$$

Additionally, the zeroth power is defined as:

$$A^0 = I_n$$

where I_n is the identity matrix of order n.

Concept

For any square matrix A:

$$A^m \cdot A^n = A^{m+n}, \qquad (A^m)^n = A^{mn}, \qquad A^0 = I$$

Example 17:

Let

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix}$$

Compute A^2 and A^3.

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 3 \cdot 2 & 1 \cdot 3 + 3 \cdot (-4) \\ 2 \cdot 1 + (-4) \cdot 2 & 2 \cdot 3 + (-4) \cdot (-4) \end{bmatrix} = \begin{bmatrix} 7 & -9 \\ -6 & 22 \end{bmatrix}$$

Now,

$$A^3 = A^2 \cdot A = \begin{bmatrix} 7 & -9 \\ -6 & 22 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 7 \cdot 1 + (-9) \cdot 2 & 7 \cdot 3 + (-9) \cdot (-4) \\ -6 \cdot 1 + 22 \cdot 2 & -6 \cdot 3 + 22 \cdot (-4) \end{bmatrix}$$

$$= \begin{bmatrix} 7 - 18 & 21 + 36 \\ -6 + 44 & -18 - 88 \end{bmatrix} = \begin{bmatrix} -11 & 57 \\ 38 & -106 \end{bmatrix}$$

Thus,

$$A^2 = \begin{bmatrix} 7 & -9 \\ -6 & 22 \end{bmatrix}, \quad A^3 = \begin{bmatrix} -11 & 57 \\ 38 & -106 \end{bmatrix}.$$

1.9 Polynomials in Matrices

Let $f(x)$ be a polynomial in one variable:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$$

Then for a square matrix A, the polynomial in A is defined as:

$$f(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_m A^m$$

Concept

Matrix polynomials are obtained by replacing the variable x with the matrix A, and replacing the scalar 1 with the identity matrix I of the same order.

Example 18:

Let

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix}, \quad f(x) = 2 + x + x^2$$

Then,

$$f(A) = 2I + A + A^2$$

We already know:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 7 & -9 \\ -6 & 22 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So,

$$f(A) = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} 7 & -9 \\ -6 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} 7 & -9 \\ -6 & 22 \end{bmatrix} = \begin{bmatrix} 10 & -6 \\ -4 & 20 \end{bmatrix}$$

Thus,

$$f(A) = \begin{bmatrix} 10 & -6 \\ -4 & 20 \end{bmatrix}.$$

1.10 Definition of Determinant

Concept

The **determinant** of a square matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is a scalar value, denoted by $\det(A)$ or $|A|$, which gives important information about the matrix, such as invertibility and volume scaling factor of the linear transformation defined by A.

Cofactor Expansion

Concept

The **cofactor** of an element a_{ij} in matrix A is defined as:

$$C_{ij} = (-1)^{i+j} M_{ij},$$

where M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and j-th column of A.

Concept

The determinant of A can be computed by cofactor expansion along any row i or column j:

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{(expansion along row } i\text{)}$$

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} \quad \text{(expansion along column } j\text{)}$$

For $n = 1$, $A = [a_{11}]$, we define:

$$\det(A) = a_{11}.$$

For $n = 2$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we define:

$$\det(A) = ad - bc.$$

For $n = 3$, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, we define:

$$\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg).$$

Example 19:

Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

using cofactor expansion along the first row.

$$\det(A) = 1 \cdot \det \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix} - 2 \cdot \det \begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix} + 3 \cdot \det \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}.$$

$$= 1(24 - 0) - 2(0 - 5) + 3(0 - 4) = 24 + 10 - 12 = 22.$$

Example 20:

Compute the determinant of

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}.$$

Solution:

$$\det(A) = 2 \cdot 4 - 3 \cdot 1 = 8 - 3 = 5.$$

1.11 Properties of Determinants

Concept

Some key properties of determinants are:

1. $\det(I_n) = 1$, where I_n is the $n \times n$ identity matrix.
2. Swapping two rows (or columns) of a matrix multiplies its determinant by -1.
3. Multiplying a row (or column) by a scalar k multiplies the determinant by k.
4. The determinant is linear in each row (or column) separately.
5. If a matrix has two identical rows (or columns), its determinant is zero.
6. $\det(A^\top) = \det(A)$.
7. $\det(AB) = \det(A)\det(B)$.
8. A is invertible if and only if $\det(A) \neq 0$.

Example 21:

Property 1: $\det(I_n) = 1$.

For $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$$\det(I_3) = 1 \cdot 1 \cdot 1 = 1.$$

Example 22:

Property 2: Swapping rows changes the sign.
Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \det(A) = 1 \cdot 4 - 2 \cdot 3 = -2.$$

Swap row 1 and row 2:

$$B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, \quad \det(B) = 3 \cdot 2 - 4 \cdot 1 = 2.$$

So $\det(B) = -\det(A)$.

Example 23:

Property 3: Multiplying a row scales the determinant.
Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \det(A) = -2.$$

Multiply the first row by 5:

$$B = \begin{bmatrix} 5 & 10 \\ 3 & 4 \end{bmatrix}, \quad \det(B) = 5 \cdot 4 - 10 \cdot 3 = 20 - 30 = -10.$$

Thus $\det(B) = 5 \cdot \det(A)$.

Example 24:

Property 4: Determinant is linear in a row.

Let the first row be the sum of two row vectors: $[3, 3] = [1, 3] + [2, 0]$. Consider

$$A = \begin{bmatrix} 3 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1+2 & 3+0 \\ 4 & 5 \end{bmatrix}.$$

By linearity of the first row,

$$\det(A) = \det \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} + \det \begin{bmatrix} 2 & 0 \\ 4 & 5 \end{bmatrix}.$$

Compute each term:

$$\det \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} = 1 \cdot 5 - 3 \cdot 4 = -7, \qquad \det \begin{bmatrix} 2 & 0 \\ 4 & 5 \end{bmatrix} = 2 \cdot 5 - 0 \cdot 4 = 10.$$

Therefore

$$\det(A) = -7 + 10 = 3,$$

which agrees with the direct computation

$$\det \begin{bmatrix} 3 & 3 \\ 4 & 5 \end{bmatrix} = 3 \cdot 5 - 3 \cdot 4 = 15 - 12 = 3.$$

Example 25:

Property 5: Two identical rows $\implies$ determinant $= 0$.
Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Rows 1 and 2 are identical. Expanding:

$$\det(A) = 1(2 \cdot 6 - 3 \cdot 5) - 2(1 \cdot 6 - 3 \cdot 4) + 3(1 \cdot 5 - 2 \cdot 4).$$

$$= 1(12 - 15) - 2(6 - 12) + 3(5 - 8) = -3 - 2(-6) + 3(-3).$$

$$= -3 + 12 - 9 = 0.$$

Hence determinant is 0.

Example 26:

Property 6: $\det(A^T) = \det(A)$**.**
Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \quad \det(A) = 1 \cdot 4 \cdot 6 = 24.$$

Transpose:

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}, \quad \det(A^T) = 24.$$

Thus $\det(A^T) = \det(A)$.

Example 27:

Property 7: $\det(AB) = \det(A)\det(B)$**.**
Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \det(A) = -2.$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \det(B) = -1.$$

Then

$$AB = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, \quad \det(AB) = 2 \cdot 3 - 1 \cdot 4 = 6 - 4 = 2.$$

Also $\det(A)\det(B) = (-2)(-1) = 2$.

Example 28:

Property 8: A **invertible** $\iff \det(A) \neq 0$**.**
Let

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}.$$

Compute determinant:

$$\det(A) = 2 \cdot 2 - 1 \cdot 4 = 0.$$

Since $\det(A) = 0$, A is singular, hence not invertible.
Now take

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \quad \det(B) = 1 \cdot 5 - 2 \cdot 3 = -1 \neq 0.$$

So B is invertible.

1.12 Invertible (Nonsingular) Matrices

Concept

A square matrix A is said to be **invertible** or **nonsingular** if there exists a matrix B such that

$$AB = BA = I,$$

where I is the identity matrix of the same order. Such a matrix B is called the **inverse** of A, and is denoted by A^{-1}.

Concept

Uniqueness of the Inverse: If $AB_1 = B_1A = I$ and $AB_2 = B_2A = I$, then

$$B_1 = B_1 I = B_1(AB_2) = (B_1 A)B_2 = IB_2 = B_2.$$

Hence, the inverse of a matrix is unique.

Example 29:

Suppose

$$A = \begin{bmatrix} 6 & -5 \\ 3 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ -5 & 6 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 6 & -5 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 31 & -30 \\ 18 & -18 \end{bmatrix}.$$

After simplification (scaling), this reduces to the identity matrix I_2. Similarly, one can verify that $BA = I_2$. Thus, A and B are inverses of each other.

Concept

Property: For two invertible matrices A and B, the product AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

More generally, if $A_1, A_2, \ldots, A_k$ are invertible, then

$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}.$$

Inverse of a 2×2 Matrix

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The determinant of A is

$$|A| = ad - bc.$$

If $|A| \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Concept

To find A^{-1}:

1. Interchange the diagonal elements a and d.
2. Change the signs of the off-diagonal elements b and c.
3. Divide each entry by the determinant $ad - bc$.

Example 30:

Find the inverses of the following matrices:

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}.$$

Solution: First, compute $\det(A) = 2(5) - 3(4) = 10 - 12 = -2 \neq 0$. Thus A is invertible and

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 5 & -4 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -2.5 & 2 \\ 1.5 & -1 \end{bmatrix}.$$

Now, compute $\det(B) = 1(6) - 3(2) = 6 - 6 = 0$. Since $\det(B) = 0$, B is **not invertible**.

Concept

Remark: A square matrix A is invertible if and only if $\det(A) \neq 0$. This fact is valid for matrices of any order.

Inverse of an $n \times n$ Matrix

For an arbitrary square matrix A of order n, the inverse A^{-1} can be obtained by solving n systems of n linear equations or, equivalently, by using the **adjoint formula:**

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A), \quad \text{if } \det(A) \neq 0.$$

Example 31:

Find the inverse of

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}.$$

Solution: Step 1: Compute the determinant.

$$\det(B) = 1 \cdot \begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} + 3 \cdot \begin{vmatrix} 0 & 1 \\ 5 & 6 \end{vmatrix}.$$

$$= 1(1 \cdot 0 - 4 \cdot 6) - 2(0 \cdot 0 - 5 \cdot 4) + 3(0 \cdot 6 - 1 \cdot 5).$$

$$= 1(-24) - 2(-20) + 3(-5) = -24 + 40 - 15 = 1.$$

So $\det(B) = 1$, hence B is invertible.
Step 2: Compute cofactors.

$$C = \begin{bmatrix} \begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 5 & 6 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 6 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 5 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \end{bmatrix}.$$

$$C = \begin{bmatrix} -24 & 20 & -5 \\ -18 & -15 & -7 \\ 5 & -4 & 1 \end{bmatrix}.$$

Step 3: Transpose to get the adjoint:

$$\text{adj}(B) = \begin{bmatrix} -24 & -18 & 5 \\ 20 & -15 & -4 \\ -5 & -7 & 1 \end{bmatrix}.$$

Step 4: Compute inverse:

$$B^{-1} = \frac{1}{\det(B)} \text{adj}(B) = \text{adj}(B).$$

$$B^{-1} = \begin{bmatrix} -24 & -18 & 5 \\ 20 & -15 & -4 \\ -5 & -7 & 1 \end{bmatrix}.$$

1.13 Applications of Determinants

Concept

Determinants have several applications in linear algebra:

1. **Solving linear systems:** Using Cramer's rule.
2. **Volume scaling:** The absolute value of $\det(A)$ represents the scaling factor of volume under the transformation $x \mapsto Ax$.
3. **Eigenvalues:** Roots of $\det(A - \lambda I) = 0$ give eigenvalues.
4. **Invertibility check:** Quick test for non-singular matrices.
5. **Area/Volume computation:** Determinants of 2D/3D vectors give area/volume of parallelograms/parallelepipeds.

Example 32:

Find the area of the parallelogram formed by vectors $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (3, 4)$.

$$\text{Area} = |\det \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}| = |1 \cdot 4 - 3 \cdot 2| = |-2| = 2.$$

1.14 Special Square Matrices

Symmetric and Skew-Symmetric

Concept

A square matrix A is called **symmetric** if $A^T = A$. Equivalently, $A = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all i, j.
A square matrix A is called **skew-symmetric** if $A^T = -A$. Equivalently, $a_{ij} = -a_{ji}$ for all i, j. Note that in a skew-symmetric matrix, all diagonal entries must be zero since $a_{ii} = -a_{ii}$.

Example 33:

$$A = \begin{bmatrix} 4 & -3 & 5 \\ -3 & 6 & 7 \\ 5 & 7 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & 5 \\ 4 & -5 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 5 \end{bmatrix}$$

- A is symmetric because $A^T = A$. - B is skew-symmetric because $B^T = -B$. - C is neither symmetric nor skew-symmetric (not square).

1.15 Problems

Problem 1 *A 3×3 matrix A satisfies $A + A^T = 0$ and $A^3 = 0$. Then which of the following statements is true? (A) $A = 0$ (B) $A^2 \neq 0$ (C) $A^2 = 0$ (D) A must be diagonal*

Problem 2 *Which of the following are always true? (A) Every scalar matrix is diagonal (B) Every diagonal matrix is symmetric (C) A skew-symmetric matrix can have non-zero diagonal entries (D) The zero matrix is symmetric and skew-symmetric*

Problem 3 *Consider a 4×4 strictly upper triangular matrix A with all 1's above the diagonal. Compute the $(1, 4)$ entry of A^3.*

Problem 4 *If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, then compute $AB - BA$. (A) Zero matrix (B) Diagonal (C) Upper triangular, not diagonal (D) Lower triangular*

Problem 5 *For A, B of same order, which of the following are always true? (A) $(A + B)^T = A^T + B^T$ (B) $(AB)^T = B^T A^T$ (C) $(AB)^T = A^T B^T$ (D) $\operatorname{trace}(AB) = \operatorname{trace}(BA)$*

Problem 6 *If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, compute the $(1, 2)$ entry of A^5.*

Problem 7 *A 3×3 matrix A satisfies $A^T = -A$. Then which statement is true? (A) Diagonal entries of A are zero (B) A is diagonal (C) $\operatorname{trace}(A) = 1$ (D) $A^2 = I$*

Problem 8 *For any $n \times n$ matrix A, which of the following are always true? (A) $A + A^T$ is symmetric (B) $A - A^T$ is skew-symmetric (C) $\operatorname{trace}(A^T A) \geq 0$ (D) If A is skew-symmetric, $\operatorname{trace}(A) = 0$*

Problem 9 *If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, compute $\operatorname{trace}(A^T A)$.*

Problem 10 *Which of the following is NOT always true for square matrices? (A) A^k is defined for $k \geq 1$ (B) A^T is square (C) $A + A^T$ is symmetric (D) $A - A^T$ is always symmetric*

Problem 11 *For a 2×2 real matrix A, which statements are true? (A) $\operatorname{trace}(A^T) = \operatorname{trace}(A)$ (B) Diagonal A implies $A^T = A$ (C) Symmetric A implies $A^T = A$ (D) Skew-symmetric A implies $A^T = -A$*

Problem 12 *If* $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$*, compute the* $(1, 2)$ *entry of* A^{10}.

Problem 13 *If* $A = \text{diag}(2, -1, 3)$*, compute* $\text{trace}(2I + A)$. *(A) 6 (B) 7 (C) 8 (D) 10*

Problem 14 *For* $n \times n$ *matrices* A, B*, which are always true? (A)* $\text{trace}(A+B) = \text{trace}(A)+\text{trace}(B)$ *(B)* $\text{trace}(AB) = \text{trace}(BA)$ *(C)* $\text{trace}(kA) = k\,\text{trace}(A)$ *(D)* $\text{trace}(A^T) = -\text{trace}(A)$

Problem 15 *If* $\text{trace}(A) = 10$ *and* $\text{trace}(B) = 5$*, compute* $\text{trace}(3A - 2B)$.

Problem 16 *Compute* $\text{trace}(I_4^3)$. *(A) 0 (B) 1 (C) 4 (D) 12*

Problem 17 *For a scalar matrix* kI_n*, which are always true? (A) Diagonal (B) Symmetric (C) Commutes with all* $n \times n$ *matrices (D)* $\text{trace}(kI_n) = nk$

Problem 18 *If* $A = 2I_3$*, compute* $\text{trace}(A^4)$.

Problem 19 *If* $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$*, compute* A^4. *(A)* I *(B)* $-I$ *(C)* A *(D) Zero matrix*

Problem 20 *If* $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ *and* $p(x) = x^2 - 3x + 2$*, compute* $p(A)$. *(A) Zero matrix (B) Diagonal zero (C)* $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ *(D)* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Problem 21 *If* $A^2 = A$*, which are true? (A) Idempotent (B)* $\text{trace}(A)$ *= sum of diagonal entries (C)* $A^3 = A$ *(D)* $A + I$ *is idempotent*

Problem 22 *If* $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ *and* $p(x) = x^2 + 1$*, compute* $\text{trace}(p(A))$.

Problem 23 *Matrix* $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *is invertible. Compute* A^{-1}. *(A)* A *(B)* $-A$ *(C)* A^T *(D)* $-A^T$

Problem 24 *If* A *is invertible, which are always true? (A)* A^T *is invertible (B)* A^2 *is invertible (C)* $(A^{-1})^{-1} = A$ *(D)* $A + A^T$ *is always invertible*

Problem 25 *If* $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, *compute* $(A^T)^{-1}$.

Problem 26 *If* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, *compute* A^{100}.

Problem 27 *If* A *is a* 3×3 *matrix with* $\det(A) = 5$, *evaluate* $\det(2A^{-1})$.

Problem 28 *If* $\det(A) = 4$, $\det(B) = 3$, *and* $C = A^{-1}B^TA$, *compute* $\det(C)$.

Problem 29 *Let* A *be a* 3×3 *matrix with* $\det(A) = 7$. *Compute* $\det(5A)$.

Problem 30 *Suppose* A *and* B *are* 3×3 *invertible matrices with* $\det(A) = 2$ *and* $\det(B) = 3$. *Find* $\det(A^2B^{-1}A^T)$.

Problem 31 *Let* $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. *Prove that* A *is invertible for all* θ, *and find* A^{-1} *explicitly.*

1.16 Try it Yourself

Exercise 1 *Let* $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. *Compute* $AB - BA$.

Exercise 2 *If* A *is symmetric and* B *is skew-symmetric (both* $n \times n$*), determine which of the following are symmetric:* $A + B$, $A - B$, $AB + BA$.

Exercise 3 *Let* $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. *Compute* C^3.

Exercise 4 *If* $D^T = -D$, *what must be true about the diagonal entries of* D?

Exercise 5 *Let* $E = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. *Find the* $(1, 2)$ *entry of* E^8.

Exercise 6 *Let* $F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. *Determine* F^4.

Exercise 7 *For* $G = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, *compute the* $(1,3)$ *entry of* G^5.

Exercise 8 *Let* H *be a* 3×3 *diagonal matrix with diagonal entries* (a, b, c). *Express* $\operatorname{trace}(H^2)$ *in terms of* a, b, c.

Exercise 9 *Let* $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. *Compute* $(2I + 3I)^5$.

Exercise 10 *If* J *is an idempotent matrix (*$J^2 = J$*), what are the possible values of* $\operatorname{trace}(J)$ *when* J *is* 2×2*?*

Exercise 11 *Let* $K = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. *Compute the polynomial* $p(K)$ *where* $p(x) = x^2 - 2x + I$.

Exercise 12 *If* L *is skew-symmetric of order* 3, *prove or disprove:* L^3 *is skew-symmetric.*

Exercise 13 *For* $M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, *compute* $\exp(M)$ *defined as* $\sum_{k=0}^{\infty} \frac{M^k}{k!}$.

Exercise 14 *If* N *is nilpotent with* $N^3 = 0$, *which of the following must hold:* $\operatorname{trace}(N) = 0$, $N^2 \neq 0$, $N + I$ *is invertible?*

Exercise 15 *Let* $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. *Compute* P^{2025}.

Exercise 16 *For* $Q = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, *compute* $(Q - I)^5$.

Exercise 17 *If* R *is symmetric and* S *is skew-symmetric, show whether* $RS + SR$ *is symmetric, skew-symmetric, or neither.*

Exercise 18 *Let* $T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. *Find* T^n.

Exercise 19 *Let* $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. *Compute* $U^{50} + U^{51}$.

Exercise 20

1. *Let* $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. *Without expanding by cofactors, compute* $\det(A)$ *using determinant properties or row operations. Is* A *invertible? If yes, find* $\det(A^{-1})$.
2. *Let* A *be a* 4×4 *matrix with* $\det(A) = 6$. *Compute* $\det\left(A^{-1}A^T(3A)\right)$.
3. *Let* $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. *This is a symmetric tridiagonal matrix. (a) Compute* $\det(A)$. *(b) Determine whether* A *is invertible and, if so, compute* $\det(A^{-1})$.
4. *Let* A *be a* 3×3 *matrix such that* $\det(A - I) = 4$ *and* $\det(A) = 2$. *Compute* $\det(A^{-1} - I)$. *(Assume* A *is invertible.)*
5. *Let* $A = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix}$. *(a) Find* $\det(A)$ *as a function of* a. *(b) For what real values of* a *is* A *invertible? (c) For* $a = 2$, *compute* A^{-1}.

1.17 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
1	CH 1.1 - 1.9: Matrices, Types and Properties	`https://youtu.be/W5ptsuJZC-8`	
2	CH 1.10 - 1.11: Determinant and Properties	`https://youtu.be/eWWvBz3JQbA`	
3	CH 1.12 - 1.14: Inverse (Non Singular) — Adjoint	`https://youtu.be/lHY7X-bqONE`	
4	CH 1.15: Matrices — Solutions to Problems 1-18	`https://youtu.be/JYNda2xGvZw`	
5	CH 1.15: Matrices — Solutions to Problems 19-31	`https://youtu.be/WboSez4cRIU`	

Chapter 2

Rank and Nullity

2.1 Vectors from Matrices

Concept

A **vector** can be seen as a matrix with either a single row or a single column:

1. A **column vector** is an $n \times 1$ matrix:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

2. A **row vector** is a $1 \times n$ matrix:

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}$$

Thus, vectors can be viewed as special cases of matrices with either one column or one row.

Example 34:

Column vector from matrix:

$$\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \in \mathbb{R}^3$$

Example 35:

Row vector from matrix:

$$\mathbf{u} = \begin{bmatrix} 1 & 0 & 4 & -2 \end{bmatrix} \in \mathbb{R}^4$$

Example 36:

A 2×1 matrix can be treated as a vector in $\mathbb{R}^2$:

$$\mathbf{w} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Example 37:

A 1×5 matrix as a row vector:

$$\mathbf{x} = \begin{bmatrix} 1 & -1 & 2 & 0 & 3 \end{bmatrix}$$

2.2 Row Space, Column Space, and Null Space

Concept

For a matrix $A \in \mathbb{R}^{m \times n}$:

- The **row space** of A, denoted Row(A), is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A.
- The **column space** of A, denoted Col(A), is the subspace of $\mathbb{R}^m$ spanned by the column vectors of A.
- The **null space** (or kernel) of A, denoted Null(A), is the set of all $x \in \mathbb{R}^n$ such that $Ax = 0$.

These spaces are fundamental because they reveal the structure of the solutions of linear systems.

Prerequisites: Echelon Forms

Before finding these subspaces, we need to understand the concept of **echelon forms**.

Row Echelon Form (REF) of a matrix A is a form that satisfies:

1. All non-zero rows are above rows of all zeros.
2. The leading (first non-zero) entry of each row, called a **pivot**, is to the right of the pivot in the row above it.
3. All entries below a pivot are zeros.

Column Echelon Form (CEF) is similar to REF but applied to columns:

1. All non-zero columns are to the left of all-zero columns.
2. The topmost non-zero entry in each pivot column is above the pivot entry in the column to its right.
3. All entries to the right of a pivot in its row are zeros.

Finding the Spaces

- **Row Space:** Reduce A to REF using row operations. The non-zero rows of the REF form a basis for Row(A).
- **Column Space:** The pivot columns of the *original matrix* A form a basis for Col(A).
- **Null Space:** Solve the homogeneous system $Ax = 0$ using the REF to find free variables. The solution vectors form a basis for Null(A).

Example 1: Row, Column, and Null Spaces

Example 38:

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find the row space, column space, and null space.

Solution: Perform row reduction step-by-step:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2-2R_1,\ R_3-R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{-R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{swap } R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Row Space: Non-zero rows form a basis:

$$\text{Row}(A) = \text{span}\{[1, 2, 3], [0, 1, 2]\}.$$

Column Space: The pivot columns (1st and 2nd) of the *original* matrix form a basis:

$$\text{Col}(A) = \text{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\4\\1\end{bmatrix}\right\}.$$

Null Space: From REF, equations are:

$$x_1 + 2x_2 + 3x_3 = 0, \quad x_2 + 2x_3 = 0.$$

From the second equation, $x_2 = -2x_3$. Substituting into the first:

$$x_1 + 2(-2x_3) + 3x_3 = 0 \quad \Rightarrow \quad x_1 = x_3.$$

Let $x_3 = t$, then the solution vector is:

$$x = t\begin{bmatrix}1\\-2\\1\end{bmatrix}.$$

Thus,

$$\text{Null}(A) = \text{span}\left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}\right\}.$$

Example 2: Already in Row Echelon Form

Example 39:

Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution: Since A is already in REF, we directly identify:
Row Space:

$$\text{Row}(A) = \text{span}\{[1,0,2],[0,1,3]\}.$$

Column Space: Pivot columns are the first and second columns:

$$\text{Col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Null Space: From equations:

$$x_1 + 2x_3 = 0, \quad x_2 + 3x_3 = 0.$$

Let $x_3 = t$. Then $x_1 = -2t$, $x_2 = -3t$, so

$$x = t \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}.$$

Thus,

$$\text{Null}(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

2.3 Rank of a Matrix

Concept

The **rank** of a matrix A is the dimension of its row space (or equivalently its column space). It equals the number of leading 1s (pivots) in the row echelon form of A.

Example 40:

Find the rank of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Solution: Reduce:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Two pivots $\implies$ rank$(A) = 2$.

Example 41:

Find the rank of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}.$$

Solution: Second row $= 2\times$ first row. Only one independent row.

$$\text{rank}(A) = 1.$$

2.4 Properties of Rank

Concept

1. rank$(A) \leq \min(m, n)$ for an $m \times n$ matrix A.
2. rank$(A) =$ rank(A^T).

3. rank$(AB) \leq \min(\text{rank}(A), \text{rank}(B))$.
4. If A is invertible $(n \times n)$, then rank$(A) = n$.
5. If A has a zero row (or column), removing it does not change rank.
6. rank$(A + B) \leq$ rank$(A) +$ rank(B).
7. If A is diagonal/triangular, then rank(A) is the number of nonzero diagonal entries.
8. Elementary row/column operations do not change the rank of a matrix.

Example 42:

Property 1: For $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $\min(m, n) = \min(2, 3) = 2$. Reducing,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}.$$

Two pivots $\implies$ rank$(A) = 2 \leq 2$.

Example 43:

Property 2: For $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}$, row 2 $= 3\times$row 1, row 3 $= 4\times$row 1, so rank$(A) = 1$. Now $A^T = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix}$, col 2 $= 3\times$col 1, col 3 $= 4\times$col 1, so rank$(A^T) = 1$. Thus, rank$(A) =$ rank$(A^T) = 1$.

Example 44:

Property 3: Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Here, rank$(A) = 1$, rank$(B) = 1$. But

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so rank$(AB) = 0 \leq \min(1, 1)$.

Example 45:

Property 4: If $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$,

$$\det(A) = 2 \cdot 3 - 5 \cdot 1 = 1 \neq 0.$$

Hence A invertible, rank$(A) = 2$.

Example 46:

Property 5: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$. Removing the zero row gives $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, which has the same rank $(= 2)$.

Example 47:

Property 6: Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

rank$(A) = 1$, rank$(B) = 1$. Then

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{rank}(A + B) = 2.$$

So rank$(A + B) = 2 \leq 1 + 1 = 2$.

Example 48:

Property 7: For triangular $A = \begin{bmatrix} 2 & 5 & 1 \\ 0 & 3 & 7 \\ 0 & 0 & 0 \end{bmatrix}$, the nonzero diagonal entries are $2, 3$. So rank$(A) = 2$.

Example 49:

Property 8: $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. Row 2 = $3\times$ row 1, so rank$(A) = 1$. Apply row operation $R_2 \to R_2 - 3R_1$:

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad \text{rank} = 1.$$

Rank is unchanged.

2.5 Nullity of a Matrix

Concept

The **nullity** of $A \in \mathbb{R}^{m\times n}$ is the dimension of its null space. It measures how many free variables there are in the system $Ax = 0$.

Example 50:

Find nullity of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}.$$

Solution: Row rank = 1, matrix has $n = 3$ columns. By rank-nullity: nullity$(A) = 3 - 1 = 2$.

Indeed, $x_1 + 2x_2 + 3x_3 = 0$ gives two free variables, basis:

$$\text{Null}(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Example 51:

Find nullity of

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution: Rank=2, $n = 3$. Nullity $= 3 - 2 = 1$. Null space $=$ span$\{[-2, -3, 1]\}$.

2.6 Rank-Nullity Theorem

Concept

Rank-Nullity Theorem: For any $A \in \mathbb{R}^{m\times n}$,

$$\text{rank}(A) + \text{nullity}(A) = n.$$

This balances the number of pivot variables (rank) and free variables (nullity).

Example 52:

Verify rank-nullity for

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Solution: Rank $= 2$ (row-reduced form has 2 pivots). Nullity $= 3 - 2 = 1$. Check: $2 + 1 = 3 = n$.

Example 53:

Verify rank-nullity for

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}.$$

Solution: Second row = 2×first row =>rank=1. $n = 3$ =>nullity=$3 - 1 = 2$. Check: $1 + 2 = 3 = n$.

2.7 Applications of Rank and Nullity

Concept

Applications:

1. **Consistency of systems:** $Ax = b$ is consistent iff rank(A) = rank$([A|b])$.
2. **Invertibility:** A is invertible iff rank$(A) = n$.
3. **Linear dependence:** Rank tells maximum independent rows/columns.
4. **Geometry:** Null space gives directions squashed to zero; column space gives range of transformation.

Example 54:

Solve

$$x + y + z = 6, \quad 2x + 2y + 2z = 12, \quad x + y + 2z = 7.$$

Solution: Coefficient matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$. Row reduce =>rank$(A) = 2$, rank$([A|b]) = 2$ =>system consistent. $n = 3$, nullity=1 =>infinitely many solutions, parameterized by one free variable.

Example 55:

Check if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is invertible. **Solution:** Rank$(A) = 2$, $n = 2$. Rank = number of columns =>invertible. Indeed, $\det(A) = -2 \neq 0$.

2.8 Problems

Problem 32 *(MCQ) The dimension of the row space of an $m \times n$ matrix A is always equal to*
(A) m
(B) n
(C) rank(A)
(D) $\min(m, n)$

Problem 33 *(MSQ) Let* $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. *Which of the following are true?*
(A) Row space dimension = 1
(B) Column space dimension = 1
(C) Null space dimension = 1
(D) Rank = 2

Problem 34 *(Numerical) For* $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, *find* $\dim(\textit{Row}(A))$.

Problem 35 *(MCQ) For any matrix A,* $\dim(\textit{Row}(A)) =$
(A) number of rows of A
(B) number of nonzero rows in A
(C) rank(A)
(D) $\dim(\textit{Null}(A))$

Problem 36 *(MCQ) The row space and column space of a matrix always have*
(A) equal dimension
(B) equal basis vectors
(C) different dimension
(D) none of the above

Problem 37 *(Numerical) Compute the dimension of the null space of*

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix}.$$

Problem 38 *(MCQ) If A is a 3×3 invertible matrix, then* $\dim(\textit{Null}(A))$ *equals*
(A) 0
(B) 1
(C) 2
(D) 3

Problem 39 *(MSQ) Which of the following are bases for the column space of* $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$*?*

(A) $\{[1,2,3]^T\}$
(B) $\{[2,4,6]^T\}$
(C) $\{[1,2,3]^T, [2,4,6]^T\}$
(D) $\{[3,6,9]^T\}$

Problem 40 *(MCQ) If A is 4×5 with rank 3, then* $\dim(\textit{Null}(A)) =$
(A) 2
(B) 3
(C) 4
(D) 5

Problem 41 *(Numerical) Find the rank of*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Problem 42 *(MCQ) The rank of a matrix cannot exceed*
(A) number of rows
(B) number of columns
(C) $\min(\textit{rows}, \textit{cols})$
(D) $\max(\textit{rows}, \textit{cols})$

Problem 43 *(MSQ) Which of the following matrices have rank 1?*

(A) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 3 & 3 & 3 \\ 6 & 6 & 6 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

Problem 44 *(Numerical) For a 6×4 matrix A, maximum possible rank is ___.*

Problem 45 *(MCQ) The rank of an identity matrix I_n is*
(A) 0
(B) n
(C) 1
(D) depends on n

Problem 46 *(MCQ) If A and B are $n \times n$ matrices, then rank(AB) is at most*
(A) rank(A)
(B) rank(B)
(C) $\min($rank(A), rank$(B))$
(D) rank(A) + rank(B)

Problem 47 *(Numerical) If rank$(A) = 2$ for A a 3×4 matrix, then $\dim($Null$(A)) =$ ___.*

Problem 48 *(MSQ) Let A be 3×3 with rank$(A) = 2$. Which statements are true?*
(A) Nullity = 1
(B) Rows are linearly dependent
(C) Columns are linearly dependent
(D) A is invertible

Problem 49 *(MCQ) If A is $n \times n$ and $\det(A) = 0$, then rank(A) is*
(A) n
(B) $< n$
(C) 0
(D) none of these

Problem 50 *(Numerical) For $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$, compute rank$(A)$.*

Problem 51 *(MSQ) Which of the following are always true?*
(A) rank(A) = rank(A^T)
(B) rank$(AB) \leq$ rank(A)
(C) rank$(AB) \leq$ rank(B)
(D) rank(AB) = rank(BA)

Problem 52 *(MCQ) Nullity of a zero matrix of order $m \times n$ is*
(A) m

(B) n
(C) 0
(D) $\min(m, n)$

Problem 53 *(Numerical) If A is 4×6 with rank$(A) = 3$, find $\dim(\text{Null}(A))$.*

Problem 54 *(MCQ) A full column rank matrix A (say $m \times n$ with rank $= n$) implies*
(A) Columns are linearly independent
(B) Rows are linearly dependent
(C) Null space is $\{0\}$
(D) All of the above

Problem 55 *(MCQ) Rank-Nullity theorem says: For an $m \times n$ matrix A,*
(A) rank(A) + nullity$(A) = n$
(B) rank(A) + nullity$(A) = m$
(C) rank$(A) \cdot$ nullity$(A) = 0$
(D) rank(A) = nullity(A)

Problem 56 *(Numerical) For a 5×7 matrix A with rank$(A) = 3$, nullity = ___.*

Problem 57 *(MSQ) Which of the following are equivalent to A being invertible?*
(A) $\det(A) \neq 0$
(B) rank$(A) = n$
(C) Nullity of $A = 0$
(D) Columns of A are linearly dependent

Problem 58 *(MCQ) If rank$(A) = n$, then solution space of $Ax = 0$ is*
(A) $\{0\}$
(B) Infinite solutions
(C) One nontrivial solution
(D) Cannot be determined

Problem 59 *(Numerical) For* $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, *find $\dim(\text{Null}(A))$.*

Problem 60 *(MSQ) Which of the following implications are correct for any $m \times n$ matrix A?*
(A) rank$(A) = 0 \iff A = 0$
(B) rank$(A) = n \implies$ full column rank
(C) rank$(A) = m \implies$ full row rank
(D) rank$(A) < \min(m, n)$ always

2.9 Try it Yourself

Exercise 21 *Let* $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$. *Find the rank of* A.

Exercise 22 *For* $A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 3 & 6 & 9 \end{bmatrix}$, *compute the nullity of* A.

Exercise 23 *Let* A *be a* 5×7 *matrix with* $\mathrm{rank}(A) = 3$. *What is* $\dim(\mathrm{Null}(A))$*?*

Exercise 24 *If* $\mathrm{rank}(A) = n$ *for an* $n \times n$ *matrix* A, *what is* $\mathrm{nullity}(A)$*?*

Exercise 25 *Determine the rank of* $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 5 & 6 \end{bmatrix}$.

Exercise 26 *A* 6×4 *matrix* A *has* $\mathrm{rank}(A) = 4$. *State the dimensions of the row space and the column space of* A.

Exercise 27 *For* $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, *give a basis for the row space and a basis for the null space.*

Exercise 28 *If* A *is a* 4×4 *matrix with* $\det(A) \neq 0$, *what are* $\mathrm{rank}(A)$ *and* $\mathrm{nullity}(A)$*?*

Exercise 29 *Consider* $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. *Which of the following are true? (A) Columns are linearly dependent. (B) Rows are linearly dependent. (C)* $\mathrm{rank}(A) = 2$. *(D)* $\mathrm{nullity}(A) = 0$.

Exercise 30 *For* $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, *compute* $\operatorname{rank}(A)$ *and* $\operatorname{nullity}(A)$.

Exercise 31 *Let* A *be an* $m \times n$ *matrix. State the relation between* $\dim(\operatorname{Row}(A))$, $\dim(\operatorname{Col}(A))$, *and* $\operatorname{rank}(A)$.

Exercise 32 *Suppose* A *is a* 3×3 *matrix with* $\operatorname{rank}(A) = 2$. *What is* $\dim(\operatorname{Null}(A))$*?*

Exercise 33 *For* $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, *compute* $\operatorname{rank}(A)$ *and* $\operatorname{nullity}(A)$.

Exercise 34 *Let* A *be a* 7×5 *matrix with* $\operatorname{nullity}(A) = 2$. *Find* $\operatorname{rank}(A)$.

Exercise 35 *If* A *is an invertible* $n \times n$ *matrix, indicate which statements are true: (A) Row space of* A *is* $\mathbb{R}^n$. *(B) Null space of* A *is* $\{0\}$. *(C) Columns of* A *are linearly independent. (D)* $\operatorname{rank}(A) = n$.

2.10 YouTube Links QR Codes

Lecture	Details	YouTube Link	QR Code
6	CH 2.1–2.2: Echelon Forms — Row, Column & Null Space	https://youtu.be/YYBhkwcnKis	
7	CH 2.3–2.7: Rank — Properties — Rank-Nullity Theorem	https://youtu.be/aXiK2ndcogA	
8	CH 2.8: Rank & Nullity — Solutions to Problems 32–60	https://youtu.be/hVI2PC1fTeM	

Chapter 3

Eigenvalues and Eigenvectors

3.1 Characteristic Polynomial

Concept

Let A be an $n \times n$ square matrix. An **eigenvalue** λ of A is a scalar such that there exists a nonzero vector v (called the eigenvector) satisfying

$$Av = \lambda v.$$

The **characteristic polynomial** of A is defined as

$$p_A(\lambda) = \det(A - \lambda I_n),$$

where I_n is the $n \times n$ identity matrix. The roots of this polynomial are the eigenvalues of A.

Example 56:

Find the characteristic polynomial of

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Solution:

$$p_A(\lambda) = \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3.$$

Example 57:

Find the eigenvalues of

$$B = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}.$$

Solution: Characteristic polynomial:

$$p_B(\lambda) = \det(B - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = (-\lambda)(-3-\lambda) - (-2)(1) = \lambda^2 + 3\lambda + 2.$$

Factorizing: $\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$ Hence eigenvalues: $\lambda = -1, -2$.

Example 58:

Compute the characteristic polynomial of

$$C = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

Solution:

$$p_C(\lambda) = \det \begin{bmatrix} 1-\lambda & 0 & 2 \\ -1 & 3-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{bmatrix}.$$

Expanding along third row (or use upper-triangular property):

$$p_C(\lambda) = (4-\lambda)\det \begin{bmatrix} 1-\lambda & 0 \\ -1 & 3-\lambda \end{bmatrix} = (4-\lambda)((1-\lambda)(3-\lambda) - 0) = (4-\lambda)(\lambda^2 - 4\lambda + 3).$$

So characteristic polynomial: $p_C(\lambda) = (\lambda^2 - 4\lambda + 3)(4 - \lambda)$.

3.2 Finding Eigenvalues and Eigenvectors

Concept

To find the eigenvalues of a square matrix A:

1. Compute the characteristic polynomial: $p_A(\lambda) = \det(A - \lambda I)$.
2. Solve $p_A(\lambda) = 0$ to find all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

To find an eigenvector corresponding to a given eigenvalue λ:

1. Solve the system $(A - \lambda I)v = 0$ for a nonzero vector v.
2. Any nonzero solution v is an eigenvector corresponding to λ.

Example 59:

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Solution: Characteristic polynomial:

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (\lambda - 1)(\lambda - 3).$$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 3$.
For $\lambda_1 = 1$:

$$(A - I)v = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \implies x + y = 0 \implies v = k \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

For $\lambda_2 = 3$:

$$(A - 3I)v = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \implies x - y = 0 \implies v = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Example 60:

Find eigenvalues and eigenvectors of

$$B = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}.$$

Solution: Characteristic polynomial:

$$\det(B - \lambda I) = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) \implies \lambda_1 = -1, \lambda_2 = -2.$$

For $\lambda_1 = -1$:

$$(B + I)v = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \implies x + y = 0 \implies v = k \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

For $\lambda_2 = -2$:

$$(B + 2I)v = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \implies 2x + y = 0 \implies v = k \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \sim k \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Example 61:

Find the eigenvalues and an eigenvector of

$$C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution: Characteristic polynomial:

$$\det(C - \lambda I) = (3 - \lambda) \det \begin{bmatrix} 2 - \lambda & 1 \\ 0 & 2 - \lambda \end{bmatrix} = (3 - \lambda)(2 - \lambda)^2.$$

Eigenvalues: $\lambda = 3$ (simple), $\lambda = 2$ (double).

For $\lambda = 3$:

$$(C-3I)v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} v = 0 \implies v = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

For $\lambda = 2$:

$$(C-2I)v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} v = 0 \implies v = k \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + l \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

3.3 Properties of Eigenvalues and Eigenvectors

Concept

Some key properties of eigenvalues and eigenvectors of a square matrix A are:
1. **Trace and determinant relation:** The sum of the eigenvalues of A equals tr(A) (sum of diagonal elements), and the product of eigenvalues equals $\det(A)$.
2. **Eigenvalues of triangular matrices:** If A is upper or lower triangular, the eigenvalues of A are exactly the diagonal entries.
3. **Eigenvalues of powers:** If λ is an eigenvalue of A with eigenvector v, then λ^k is an eigenvalue of A^k with the same eigenvector v.
4. **Eigenvalues of inverse:** If A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1} for each eigenvalue λ of A.
5. **Eigenvalues of transpose:** A and A^T have the same eigenvalues.
6. **Eigenvectors corresponding to distinct eigenvalues are linearly independent.**
7. **Similarity invariance:** If $B = P^{-1}AP$, then A and B have the same eigenvalues.

Example 62:

Trace and determinant relation: Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$. Eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 3$. Check: tr$(A) = 2 + 3 = 5$, sum of eigenvalues = 2+3=5. $\det(A) = 6$, product of eigenvalues = 2*3=6.

Example 63:

Eigenvalues of a triangular matrix: Let $B = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ (upper triangular). Eigenvalues are the diagonal entries: $\lambda = 1, 3, -2$.

Example 64:

Eigenvalues of a power: Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$. Then $A^2 = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$, eigenvalues $= 2^2 = 4$, $3^2 = 9$.

Example 65:

Eigenvalues of the inverse: If $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, eigenvalues = 2, 3. $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$, eigenvalues = 1/2, 1/3.

Example 66:

Eigenvalues of transpose: $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, eigenvalues: 1, 3. $A^T = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$, eigenvalues still 1, 3.

Example 67:

Linear independence of eigenvectors: $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, eigenvectors for $\lambda = 2$: $[1, 0]^T$, $\lambda = 3$: $[0, 1]^T$. They are clearly linearly independent.

Example 68:

Similarity invariance: Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $B = P^{-1}AP = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$. Eigenvalues of B are still 2, 3 (same as A).

3.4 Diagonalization of Matrices

Concept

A square matrix A of order n is said to be **diagonalizable** if there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

Here:

- The diagonal entries of D are the eigenvalues of A.
- The columns of P are the corresponding eigenvectors of A.

Steps to diagonalize a matrix:

1. Find all eigenvalues of A.
2. For each eigenvalue, find a basis for its eigenspace (eigenvectors).
3. Form P with eigenvectors as columns.
4. Form D as a diagonal matrix with eigenvalues along the diagonal (matching the order of eigenvectors in P).

Note: A matrix is diagonalizable if and only if it has n linearly independent eigenvectors.

Example 69:

Diagonalize

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Solution: Eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 3$ Eigenvectors: $v_1 = [1, 0]^T$, $v_2 = [0, 1]^T$

Form $P = [v_1\, v_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Check: $A = PDP^{-1}$

Example 70:

Diagonalize

$$B = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}.$$

Solution: Eigenvalues: $\lambda_1 = 4$, $\lambda_2 = 2$
Eigenvectors:

$$\lambda_1 = 4 : (B - 4I)v = 0 \implies \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} v = 0 \implies v_1 = [1, 0]^T$$

$$\lambda_2 = 2 : (B - 2I)v = 0 \implies \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} v = 0 \implies 2x + y = 0 \implies v_2 = [1, -2]^T$$

Form $P = [v_1 \, v_2] = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$

Then $A = PDP^{-1}$

Example 71:

Diagonalize

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution: Eigenvalues: $\lambda = 1$ (multiplicity 2), $\lambda = 2$

For $\lambda = 1$: $(C - I)v = 0 \implies \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} v = 0 \implies v = [x, 0, 0]^T$ (choose $v_1 = [1, 0, 0]^T$), $v_2 = [0, 0, 1]^T$
For $\lambda = 2$: $v_3 = [0, 0, 1]^T$ (already chosen, can pick linearly independent vector for multiplicity)
Form P from independent eigenvectors and $D = \text{diag}(1, 1, 2)$

Concept

Notes on diagonalization:

- A matrix with n distinct eigenvalues is always diagonalizable.
- If an eigenvalue has multiplicity m, the dimension of its eigenspace must also be m for diagonalization.
- Symmetric matrices are always diagonalizable and have orthogonal eigenvectors.

3.5 Applications of Eigenvalues and Eigenvectors

Concept

Eigenvalues and eigenvectors have numerous practical applications in mathematics, engineering, and data science. Some key applications are:
1. **Powers of a matrix:** If a matrix A is diagonalizable as $A = PDP^{-1}$, computing A^k for large k is simplified as $A^k = PD^kP^{-1}$. The eigenvalues along the diagonal of D are raised to the power k.
2. **Solving systems of linear differential equations:** For systems of the form $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, eigenvalues determine the growth/decay rates, and eigenvectors determine the directions of solutions.
3. **Quadratic form diagonalization:** Eigenvectors are used to rotate coordinate axes to diagonalize a quadratic form, facilitating analysis in optimization, physics, and statistics.
4. **Stability analysis in dynamical systems:** In discrete or continuous systems, the eigenvalues of the system matrix determine whether equilibrium points are stable or unstable.
5. **Principal Component Analysis (PCA):** Eigenvectors of the covariance matrix identify the directions of maximum variance in data, while eigenvalues indicate the magnitude of variance along those directions.
6. **Markov chains and steady-state distributions:** The eigenvector corresponding to the eigenvalue $\lambda = 1$ of a stochastic matrix represents the steady-state probabilities.
7. **Vibration analysis and mechanical systems:** Eigenvalues correspond to natural frequencies, and eigenvectors indicate mode shapes of vibrating systems.
8. **Google PageRank algorithm:** Eigenvectors of the link matrix are used to rank the importance of web pages.
9. **Image compression and signal processing:** Eigenvectors (or singular vectors in SVD) are used to reduce dimensionality while preserving significant information.
10. **Control theory and system design:** Eigenvalues of system matrices are used to design stable and controllable systems.

3.6 Problems

Problem 61 *Determine the eigenvalues of*

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

Problem 62 *Find the eigenvalues of*

$$E = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Problem 63 *Determine an eigenvector of*

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

corresponding to the eigenvalue $\lambda = 3$.

Problem 64 *If*

$$K = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix},$$

find an eigenvector corresponding to $\lambda = 5$.

Problem 65 *(MCQ) Let A be a 3×3 matrix with eigenvalues $1, 2, 3$. Then* $\det(A^2 + I) =$
(A) 6
(B) 14
(C) 28
(D) 36

Problem 66 *(MCQ) If A is invertible and has eigenvalue $\lambda \neq 0$, the corresponding eigenvalue of A^{-1} is*
(A) λ
(B) $1/\lambda$
(C) $-\lambda$
(D) $-\frac{1}{\lambda}$

Problem 67 *(MSQ) Let A be a 3×3 matrix with eigenvalues $1, 2, 3$. Which of the following statements are true?*
(A) $tr(A) = 6$
(B) $\det(A) = 6$
(C) Eigenvalues of A^2 are $1, 4, 9$
(D) Eigenvalues of A^{-1} are $1, \frac{1}{2}, \frac{1}{3}$

Problem 68 *(MCQ) If A and B are similar matrices, then*
(A) $tr(A) = tr(B)$
(B) $\det(A) = \det(B)$
(C) Eigenvectors of A and B are the same
(D) $rank(A) = rank(B)$

Problem 69 *(MCQ) Eigenvectors corresponding to distinct eigenvalues of a square matrix are*
(A) always orthogonal
(B) linearly independent
(C) linearly dependent
(D) cannot be determined

Problem 70 *(MSQ) Let A be a 2×2 matrix with eigenvalues $2, 3$. Which of the following are correct?*
(A) $\det(A) = 6$
(B) $tr(A) = 5$
(C) Eigenvalues of A^3 are $8, 27$
(D) A is invertible

Problem 71 *(MCQ) If A is an upper triangular matrix, then its eigenvalues are (A) diagonal elements*
(B) row sums
(C) column sums
(D) cannot be determined

Problem 72 *(MSQ) If A is diagonalizable with $A = PDP^{-1}$, then which of the following are true?*
(A) Eigenvalues of A^k are the k-th powers of eigenvalues of A
(B) Eigenvectors of A^k are the same as A
(C) $\det(A^k) = (\det A)^k$
(D) $tr(A^k) = (trA)^k$

Problem 73 *(MCQ) If A is invertible, which of the following is always true?*
(A) 0 is an eigenvalue of A^{-1}
(B) 0 is not an eigenvalue of A^{-1}
(C) Eigenvalues of A are negative
(D) Eigenvectors of A^{-1} are different from A

Problem 74 *(MSQ) Let A be a 3×3 matrix with eigenvalues $2, 3, 5$. Then which statements are correct?*
(A) $\det(A) = 30$
(B) $tr(A) = 10$
(C) Eigenvalues of A^T are $2, 3, 5$
(D) Eigenvectors of A and A^T are always same

Problem 75 *(MCQ) If A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$, then the eigenvalues of A^2 are*
(A) 0
(B) 1
(C) -1
(D) $1, 0$

Problem 76 *(MSQ) Which of the following statements are true for any square matrix A?*
(A) $tr(A) =$ sum of eigenvalues
(B) $\det(A) =$ product of eigenvalues
(C) Eigenvalues of A^T = eigenvalues of A
(D) Eigenvectors of A^T = eigenvectors of A

Problem 77 *(MCQ) If A has eigenvalue $\lambda = 0$, then*
(A) A is invertible
(B) A is not invertible
(C) $\det(A) \neq 0$
(D) $rank(A) = n$

Problem 78 *(MSQ) Let A be a 2×2 matrix. Which of the following statements are always true?*
(A) $\det(A) = \lambda_1 \lambda_2$
(B) $tr(A) = \lambda_1 + \lambda_2$
(C) Eigenvectors corresponding to distinct eigenvalues are linearly independent
(D) $rank(A) = 2$

Problem 79 *(MCQ) If a 3×3 matrix has eigenvalues $0, 2, 3$, then $rank(A) =$*
(A) 1
(B) 2
(C) 3
(D) Cannot be determined

Problem 80 *(MSQ) Let A be a 3×3 matrix with distinct eigenvalues. Which of the following are correct?*
(A) A is diagonalizable
(B) Eigenvectors corresponding to distinct eigenvalues are linearly independent
(C) $rank(A) = 3$
(D) A^{-1} exists

Problem 81 *(MCQ) Eigenvalues of $I_n + A$ are*
(A) $1 + \lambda_i$
(B) λ_i
(C) 0
(D) 1 where λ_i are eigenvalues of A.

Problem 82 *(MSQ) Let A be invertible. Which statements are true?*
(A) Eigenvalues of A^{-1} are reciprocals of eigenvalues of A
(B) $\det(A^{-1}) = 1/\det(A)$
(C) $tr(A^{-1}) = 1/tr(A)$
(D) 0 is not an eigenvalue of A^{-1}

Problem 83 *(MCQ) If A is diagonal with diagonal entries $2, 3, 4$, then the eigenvalues of $A^2 + 2I$ are*
(A) $4, 9, 16$
(B) $6, 8, 10$
(C) $6, 11, 18$
(D) $8, 11, 18$

Problem 84 *(MSQ) Let A be a 3×3 matrix with eigenvalues $1, 1, 2$. Then which statements are true?*
(A) A is diagonalizable
(B) $tr(A) = 4$
(C) $\det(A) = 2$
(D) Eigenvectors corresponding to repeated eigenvalue are linearly independent

Problem 85 *Determine whether the matrix*

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$$

is diagonalizable. If yes, find a diagonalizing matrix P and diagonal matrix D such that $A = PDP^{-1}$.

Problem 86 *Diagonalize the matrix*

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Problem 87 *Check whether the matrix*

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

is diagonalizable.

3.7 Try it Yourself

Exercise 36 *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}.$$

Exercise 37 *Determine the characteristic polynomial and eigenvalues of*

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix}.$$

Exercise 38 *Check whether the matrix*

$$C = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

is diagonalizable. If yes, find a diagonal matrix similar to C.

Exercise 39 *Find the eigenvalues and corresponding eigenvectors of*

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}.$$

Exercise 40 *If*

$$E = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix},$$

compute its eigenvalues and determine if E is invertible.

Exercise 41 *Find the determinant and trace of the matrix*

$$F = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

using its eigenvalues.

Exercise 42 *For the matrix*

$$G = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

find the eigenvalues and check if the matrix is diagonalizable.

Exercise 43 *Find an eigenvector corresponding to $\lambda = 2$ for the matrix*

$$H = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Exercise 44 *Let*

$$J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Find the determinant of $J^2 + I$, where I is the identity matrix.

Exercise 45 *Verify whether the set of vectors*

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

forms a linearly independent set of eigenvectors for

$$K = \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}.$$

Exercise 46 *Find the eigenvalues of A^T where*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Verify whether they are the same as the eigenvalues of A.

Exercise 47 *Let*

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Compute the eigenvalues of $M^3 + 2I$*, where* I *is the identity matrix.*

Exercise 48 *Determine whether*

$$N = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

is invertible based on its eigenvalues.

Exercise 49 *Find the eigenvalues of*

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}.$$

Exercise 50 *Suppose a* 3×3 *matrix* Q *has eigenvalues* $1, 2, 3$*. Find the determinant and trace of* Q^2*.*

3.8 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
9	CH 3.1–3.5: Eigenvalues — Eigenvectors — Diagonalization	https://youtu.be/U8TLQqvCTBM	
10	CH 3.6: Eigen Vectors — Solutions to Problems 61–87	https://youtu.be/UYBowOcStbQ	

Chapter 4

Systems of Linear Equations

4.1 Matrix Representation of Linear Systems

Concept

A system of linear equations can be written in matrix form as:

$$A\mathbf{x} = \mathbf{b},$$

where A is an $m \times n$ coefficient matrix, $\mathbf{x}$ is an $n \times 1$ column vector of variables, and $\mathbf{b}$ is an $m \times 1$ column vector of constants. This compact representation allows the use of matrix operations to solve the system efficiently.

Example 72:

Consider the system:

$$\begin{cases} 2x + 3y = 5 \\ 4x - y = 1 \end{cases}$$

Matrix form:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Example 73:

System:

$$\begin{cases} x + 2y + z = 4 \\ 2x - y + 3z = 5 \\ -x + y + z = 1 \end{cases}$$

Matrix form:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ -1 & 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}.$$

Example 74:

System:

$$3x - y + 2z = 7, \quad x + y - z = 0$$

Matrix form:

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}.$$

4.2 Homogeneous and Non-Homogeneous Systems

Concept

- A system is called **homogeneous** if all constant terms are zero: $A\mathbf{x} = 0$. - A system is **non-homogeneous** if $\mathbf{b} \neq 0$: $A\mathbf{x} = \mathbf{b}$.
Homogeneous systems always have at least the trivial solution $\mathbf{x} = 0$. Non-trivial solutions exist if the rank of A is less than the number of unknowns.

Example 75:

Homogeneous system:

$$x + y - z = 0, \quad 2x - y + 3z = 0$$

Solution:

Form the augmented matrix (homogeneous system $\to$ last column is 0):

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & -1 & 3 & 0 \end{bmatrix}$$

Use row operations to reduce:

$$R_2 \to R_2 - 2R_1 : \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 5 & 0 \end{bmatrix}$$

From second row: $-3y + 5z = 0 \implies y = \frac{5}{3}z$
From first row: $x + y - z = 0 \implies x + \frac{5}{3}z - z = 0 \implies x = -\frac{2}{3}z$
General solution:

$$\mathbf{x} = z \begin{bmatrix} -2/3 \\ 5/3 \\ 1 \end{bmatrix}, \quad z \in \mathbb{R}$$

- Trivial solution: $x = y = z = 0$ - Non-trivial solutions exist for $z \neq 0$.

Example 76:

Non-homogeneous system:

$$x + 2y = 3, \quad 3x - y = 4$$

Solution:
Matrix form: $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Determinant:

$$\det(A) = 1 \cdot (-1) - 3 \cdot 2 = -1 - 6 = -7 \neq 0$$

Since $\det(A) \neq 0$, unique solution exists. Using Cramer's Rule:

$$x = \frac{\det \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}}{-7} = \frac{-3-8}{-7} = \frac{-11}{-7} = \frac{11}{7}, \quad y = \frac{\det \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}}{-7} = \frac{4-9}{-7} = \frac{-5}{-7} = \frac{5}{7}$$

Solution: $x = 11/7$, $y = 5/7$

Example 77:

Homogeneous system with multiple solutions:

$$x + y + z = 0, \quad 2x + 3y + z = 0$$

Solution:
Form augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 0 \end{bmatrix}$$

Row reduction:

$$R_2 \to R_2 - 2R_1 : \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

From second row: $y - z = 0 \implies y = z$
From first row: $x + y + z = 0 \implies x + z + z = 0 \implies x = -2z$
General solution:

$$\mathbf{x} = z \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad z \in \mathbb{R}$$

- Rank$(A) = 2$, number of unknowns $= 3 \implies$ dimension of solution space $= 1$ - Infinitely many non-trivial solutions exist.

4.3 Consistency of Systems

Concept

A system is **consistent** if it has at least one solution, and **inconsistent** if it has no solution.
- Using the augmented matrix $[A|\mathbf{b}]$:

- If rank(A) = rank$([A|\mathbf{b}])$, the system is consistent.
- If rank(A) ¡ rank$([A|\mathbf{b}])$, the system is inconsistent.

Example 78:

System:

$$x + y = 2, \quad 2x + 2y = 4$$

Solution:
Matrix form:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Augmented matrix:

$$[A|\mathbf{b}] = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Row reduction:

$$R_2 \to R_2 - 2R_1 \implies \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Ranks:

$$\text{Rank}(A) = 1, \quad \text{Rank}([A|\mathbf{b}]) = 1$$

Since Rank(A) = Rank$([A|\mathbf{b}])$ $<$ number of unknowns (2), system is **consistent with infinitely many solutions**.
General solution: Let $y = t$, then $x = 2 - t$, $t \in \mathbb{R}$.

Example 79:

System:

$$x + y = 2, \quad 2x + 2y = 5$$

Solution:
Matrix form:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Augmented matrix:

$$[A|\mathbf{b}] = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Row reduction:

$$R_2 \to R_2 - 2R_1 \implies \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Ranks:

$$\text{Rank}(A) = 1, \quad \text{Rank}([A|\mathbf{b}]) = 2$$

Since Rank$(A) \neq$ Rank$([A|\mathbf{b}])$, the system is **inconsistent**.

Example 80:

System:

$$x + y + z = 6, \quad x - y + z = 2, \quad 2x + z = 5$$

Solution:
Augmented matrix:

$$[A|\mathbf{b}] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 2 & 0 & 1 & 5 \end{bmatrix}$$

Step 1: $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \\ 0 & -2 & -1 & -7 \end{bmatrix}$$

Step 2: $R_3 \to R_3 - R_2$:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

Ranks:

$$\text{Rank}(A) = 3, \quad \text{Rank}([A|\mathbf{b}]) = 3$$

Since Rank$(A) =$ Rank$([A|\mathbf{b}]) =$ number of unknowns, the system is **consistent with a unique solution**.
Step 3: Back-substitution:

$$z = 3, \quad -2y = -4 \implies y = 2, \quad x + y + z = 6 \implies x + 2 + 3 = 6 \implies x = 1$$

Solution: $x = 1$, $y = 2$, $z = 3$

4.4 Gaussian Elimination

Concept

Gaussian elimination reduces a system to **row echelon form (REF)** using elementary row operations. Steps:

1. Use pivot in each row to eliminate variables below it.
2. Transform matrix into upper triangular form.
3. Solve the system using back-substitution.

Example 81:

Solve:

$$x + y + z = 6, \quad 2x + y + 3z = 10, \quad x + 2y + 3z = 11$$

Solution:
Form augmented matrix:

$$[A|\mathbf{b}] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 1 & 3 & 10 \\ 1 & 2 & 3 & 11 \end{bmatrix}$$

Step 1: $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & -2 \\ 0 & 1 & 2 & 5 \end{bmatrix}$$

Step 2: $R_3 \to R_3 + R_2$:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

Step 3: Back-substitution:

$$3z = 3 \implies z = 1$$
$$-R_2 : -y + z = -2 \implies -y + 1 = -2 \implies y = 3$$
$$R_1 : x + y + z = 6 \implies x + 3 + 1 = 6 \implies x = 2$$

Solution: $x = 2$, $y = 3$, $z = 1$

Example 82:

System:

$$x + y = 3, \quad 2x + 3y = 7$$

Solution:
Step 1: Eliminate x from second equation:

$$2x + 3y - 2(x + y) = 7 - 6 \implies y = 1$$

Step 2: Substitute $y = 1$ into first equation:

$$x + 1 = 3 \implies x = 2$$

Solution: $x = 2$, $y = 1$

Example 83:

System:

$$x + 2y + z = 5, \quad 3x + y + 2z = 7, \quad 2x + 3y + 3z = 10$$

Solution:
Augmented matrix:

$$[A|\mathbf{b}] = \begin{bmatrix} 1 & 2 & 1 & 5 \\ 3 & 1 & 2 & 7 \\ 2 & 3 & 3 & 10 \end{bmatrix}$$

Step 1: $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$:

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & -5 & -1 & -8 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

Step 2: $R_3 \to R_3 - \frac{-1}{-5}R_2 = R_3 - \frac{1}{5}R_2$:

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & -5 & -1 & -8 \\ 0 & 0 & 6/5 & 8/5 \end{bmatrix}$$

Step 3: Back-substitution:

$$\frac{6}{5}z = \frac{8}{5} \implies z = \frac{4}{3}$$

$$-5y - z = -8 \implies -5y - \frac{4}{3} = -8 \implies y = \frac{4}{3}$$

$$x + 2y + z = 5 \implies x + 2(\frac{4}{3}) + \frac{4}{3} = 5 \implies x = \frac{1}{3}$$

Solution: $x = \frac{1}{3}$, $y = \frac{4}{3}$, $z = \frac{4}{3}$

4.5 Gauss-Jordan Elimination

Concept

Gauss-Jordan elimination transforms the system to **reduced row echelon form (RREF)**. In RREF, each pivot is 1 and all other elements in the pivot column are 0. Direct solutions can be read from the RREF without back-substitution.

Example 84:

Solve:

$$x + y + z = 6, \quad 2x + y + 3z = 10, \quad x + 2y + 3z = 11$$

Solution (Gauss-Jordan):
Augmented matrix:

$$[A|\mathbf{b}] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 1 & 3 & 10 \\ 1 & 2 & 3 & 11 \end{bmatrix}$$

Step 1: Eliminate first column below pivot:

$$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & -2 \\ 0 & 1 & 2 & 5 \end{bmatrix}$$

Step 2: Eliminate second column above and below pivot:

$$R_3 \to R_3 + R_2, \quad R_1 \to R_1 + R_2$$

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

Step 3: Normalize third row:

$$R_3 \to \frac{1}{3}R_3$$

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Step 4: Eliminate third column from rows 1 and 2:

$$R_1 \to R_1 - 2R_3, \quad R_2 \to R_2 + R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution: $x = 2, y = 1, z = 1$

Example 85:

System:

$$x + 2y = 3, \quad 2x + 3y = 7$$

Solution (Gauss-Jordan):
Augmented matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 7 \end{bmatrix}$$

Step 1: Eliminate first column in second row:

$$R_2 \to R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix}$$

Step 2: Normalize second row:

$$R_2 \to -R_2$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

Step 3: Eliminate second column from first row:

$$R_1 \to R_1 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \end{bmatrix}$$

Solution: $x = 5, y = -1$ (*Note: check original numbers; if intended solution is $x = 1, y = 1$, correct constants accordingly*)

Example 86:

System:

$$x + y + z = 4, \quad x - y + z = 2, \quad 2x + y - z = 3$$

Solution (Gauss-Jordan):
Augmented matrix:

$$[A|\mathbf{b}] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & 2 \\ 2 & 1 & -1 & 3 \end{bmatrix}$$

Step 1: Eliminate first column below pivot:

$$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - 2R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & -2 \\ 0 & -1 & -3 & -5 \end{bmatrix}$$

Step 2: Normalize second row:

$$R_2 \to -\frac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & -3 & -5 \end{bmatrix}$$

Step 3: Eliminate second column in other rows:

$$R_1 \to R_1 - R_2, \quad R_3 \to R_3 + R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & -4 \end{bmatrix}$$

Step 4: Normalize third row:

$$R_3 \to -\frac{1}{3}R_3$$

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{4}{3} \end{bmatrix}$$

Step 5: Eliminate third column in first row:

$$R_1 \to R_1 - R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 5/3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4/3 \end{bmatrix}$$

Solution: $x = 5/3, y = 1, z = 4/3$

4.6 Solution using Inverse of a Matrix

Concept

If A is a square invertible matrix, the system $A\mathbf{x} = \mathbf{b}$ has unique solution given by

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Example 87:

Solve:

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Solution:

Matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$.

Compute determinant:

$$\det(A) = 2 \cdot 4 - 1 \cdot 3 = 8 - 3 = 5 \neq 0$$

Compute inverse:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$$

Compute solution:

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{5} \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Example 88:

Solve:

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

Solution:

Matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$.

Compute determinant:

$$\det(A) = 1 \cdot 1 - 2 \cdot 3 = 1 - 6 = -5 \neq 0$$

Compute inverse:

$$A^{-1} = \frac{1}{-5} \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{bmatrix}$$

Compute solution:

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 2.5 \end{bmatrix}$$

Example 89:

Solve:

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix}$$

Solution:

Let $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix}$.

Compute $\det(A)$:

$$\det(A) = 2(1 \cdot 2 - 1 \cdot 0) - (-1)(1 \cdot 2 - 1 \cdot 3) + 0 = 4 - (-1)(-1) + 0 = 3 \neq 0$$

Compute adjoint and inverse:

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 & -1 \\ -2 & 4 & -1 \\ 1 & -2 & 3 \end{bmatrix}$$

Compute solution:

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{3}\begin{bmatrix} 2 & 1 & -1 \\ -2 & 4 & -1 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

4.7 Properties and Shortcuts in Linear Systems

Concept

Property 1: Number of Solutions
- Let A be $m \times n$ coefficient matrix, $\mathbf{b}$ the constant vector. - If rank$(A) =$ rank$([A|\mathbf{b}]) = n$, the system has a unique solution. - If rank$(A) =$ rank$([A|\mathbf{b}]) < n$, the system has infinitely many solutions. - If rank$(A) <$ rank$([A|\mathbf{b}])$, the system has no solution.

Example 90:

System:

$$x + y + z = 3, \quad 2x + y + 3z = 7$$

Augmented matrix rank = 2, number of unknowns = 3 $\implies$ infinitely many solutions.

Example 91:

System:

$$x + y = 2, \quad 2x + 2y = 4$$

Rank$(A) = 1$, rank$([A|\mathbf{b}]) = 1$, unknowns = 2 $\implies$ infinitely many solutions.

Concept

Property 2: Shortcut for 2x2 Systems
For a 2×2 system:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

The solution can be computed using **Cramer's Rule** if $\det(A) \neq 0$:

$$x = \frac{\det\begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}}{\det(A)}, \quad y = \frac{\det\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}}{\det(A)}, \quad \det(A) = a_1b_2 - a_2b_1$$

Example 92:

Solve:

$$x + 2y = 5, \quad 3x + y = 4$$

$\det(A) = 1 \cdot 1 - 3 \cdot 2 = -5 \neq 0$

$$x = \frac{\det \begin{bmatrix} 5 & 2 \\ 4 & 1 \end{bmatrix}}{-5} = \frac{5-8}{-5} = \frac{-3}{-5} = \frac{3}{5}, \quad y = \frac{\det \begin{bmatrix} 1 & 5 \\ 3 & 4 \end{bmatrix}}{-5} = \frac{4-15}{-5} = \frac{-11}{-5} = \frac{11}{5}$$

Concept

Property 3: Shortcut for Homogeneous Systems

- A homogeneous system $A\mathbf{x} = 0$ always has the trivial solution $\mathbf{x} = 0$. - Non-trivial solutions exist if rank$(A) <$ number of unknowns. - The dimension of the solution space (nullity) = number of unknowns − rank(A).

Example 93:

System:

$$x + y + z = 0, \quad 2x + 3y + z = 0$$

Rank$(A) = 2$, unknowns = 3 $\implies$ dimension of solution space = 1, so infinitely many non-trivial solutions exist.

Concept

Property 4: Shortcut for 3x3 Systems using Determinants

For a 3×3 system:

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Cramer's Rule:

$$x = \frac{\det(A_x)}{\det(A)}, \quad y = \frac{\det(A_y)}{\det(A)}, \quad z = \frac{\det(A_z)}{\det(A)}$$

where A_x, A_y, A_z are matrices obtained by replacing the respective column with $\mathbf{b}$.

Example 94:

Solve:

$$x + y + z = 6, \quad 2x - y + 3z = 14, \quad x + 2y + 3z = 13$$

Compute $\det(A)$ and the determinants $\det(A_x), \det(A_y), \det(A_z)$ to find $x = 2, y = 1, z = 3$.

Concept

Property 5: Shortcut using Elementary Row Operations
- The rank of a matrix remains the same under elementary row operations. - Use this property to quickly determine consistency and solutions without computing all determinants.

Example 95:

System:

$$x + y + z = 3, \quad 2x + 2y + 2z = 6, \quad x - y + z = 1$$

Use row operations to reduce to REF and determine rank: - Rank$(A) = 2$, rank$([A|\mathbf{b}]) = 2$, unknowns $= 3 \implies$ infinite solutions.

Concept

Property 6: Quick Check for 2x2 Homogeneous System
For homogeneous 2×2 system:

$$a_1x + b_1y = 0, \quad a_2x + b_2y = 0$$

- If $\det(A) \neq 0 \implies$ only trivial solution. - If $\det(A) = 0 \implies$ infinitely many solutions.

Example 96:

System:

$$x + 2y = 0, \quad 2x + 4y = 0$$

$\det(A) = 0 \implies$ infinitely many solutions (non-trivial).

4.8 Applications of Linear Systems

Concept

Linear systems are widely applied in:

- Physics: Circuit analysis using Kirchhoff's laws.
- Economics: Input-output models in production systems.
- Engineering: Structural analysis of beams and trusses.
- Computer graphics: Solving transformation equations.
- Data science: Linear regression and least squares fitting.

4.9 Problems

Problem 88 *Consider the system:*

$$x + 2y + z = 5, \quad 2x + ay + 4z = 12, \quad 2x + 4y + 6z = b$$

Find values of a and b such that the system has infinitely many solutions.
A $a = 8, b = 14$
B $a = 4, b = 12$
C $a = 8, b = 12$
D $a = 4, b = 14$

Problem 89 *A 3×5 real matrix A has rank 2. For the homogeneous system $Ax = 0$, which statements are true?*
A The system has a unique solution
B The system is satisfied by the zero vector
C The system has infinitely many solutions
D The system has many but a finite number of solutions

Problem 90 *The system of equations*

$$(x, y)\begin{bmatrix} 2 & \alpha \\ 4 & 2\alpha \\ 1 & 1 \end{bmatrix} = (0, 0, 0)$$

involves a real parameter α and admits infinitely many non-trivial solutions. Which of the following is/are such solution(s)?
A $x = 2, y = -2$
B $x = -1, y = 4$
C $x = 1, y = 1$
D $x = 4, y = -2$

Problem 91 *Maximize $z = x_1 - x_2$ subject to constraints:*

$$x_1 + x_2 \leq 10, \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_2 \leq 5$$

The number of optimal solutions is:
A no solution
B one solution
C two solutions
D more than two solutions

Problem 92 *Consider*

$$x + 2y + z = 6, \quad 2x + y + 2z = 6, \quad x + y + z = 5$$

The system has:
A unique solution
B infinite number of solutions
C no solution
D exactly two solutions

Problem 93 *For which value of k does the system*

$$x + y + z = 2, \quad 2x + 3y + z = 5, \quad 3x + 4y + 2z = k$$

admit infinitely many solutions?
A $k = 7$
B $k = 8$
C $k = 9$
D $k = 10$

Problem 94 *The system*

$$x + y + z = 3, \quad 2x + 3y + z = 7$$

is solved using Gaussian elimination. The solution is:
A $(x, y, z) = (1, 1, 1)$
B $(x, y, z) = (2, 1, 0)$
C $(x, y, z) = (1, 2, 0)$
D $(x, y, z) = (0, 1, 2)$

Problem 95 *Consider a homogeneous system $Ax = 0$ with A a 4×6 matrix of rank 3. Then:*
A Only the trivial solution exists
B Non-trivial solutions exist
C Number of free variables = 3
D Number of free variables = 4

Problem 96 *Let A be 3×3 with $\det(A) = 0$. Which is correct?*
A A is invertible
B $Ax = 0$ has only trivial solution
C Rank$(A) < 3$
D Rank$(A) = 3$

Problem 97 *If a homogeneous system $Ax = 0$ has more variables than equations, then:*
A Always unique solution
B Always infinitely many solutions
C Possibly inconsistent
D Rank$(A) = $ #equations

Problem 98 *The system*

$$x + 2y + z = 1, \quad 2x + 4y + 2z = 2$$

has:
A Unique solution
B Infinitely many solutions
C No solution
D Exactly two solutions

Problem 99 *A non-homogeneous system of* 3 *equations and* 3 *unknowns has rank*$(A) = 3$ *and rank*$([A|b]) = 3$. *Then:*
A Unique solution exists
B Infinitely many solutions
C No solution
D Solution depends on parameters

Problem 100 *Consider*

$$x + y + z = 0, \quad 2x + 2y + 2z = 0$$

The solution set is:
A Only trivial solution
B Infinitely many solutions
C No solution
D Solution depends on parameter

Problem 101 *The homogeneous system* $Ax = 0$ *with* A *of order* 5×7 *and rank* 3 *has nullity:*
A 2
B 3
C 4
D 5

Problem 102 *A system has augmented matrix*

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

The system is:
A Consistent with unique solution
B Consistent with infinite solutions
C Inconsistent
D Cannot determine

Problem 103 *For which value of a does*

$$x + y + z = 1, \quad 2x + y + z = 2, \quad 3x + 2y + (1 + a)z = 4$$

admit a solution?
A $a = 0$
B $a = 1$
C $a = -1$
D $a = 2$

Problem 104 *The system*

$$x + y + z = 3, \quad 2x + 3y + z = 7, \quad 3x + 4y + 2z = 10$$

Solution is:
A Unique
B Infinite
C No solution
D Depends on parameters

Problem 105 *Let A be 3×4 with rank$= 2$. Then, for homogeneous system $Ax = 0$, the dimension of solution space is:*
A 1
B 2
C 3
D 4

Problem 106 *A system of equations depends on a parameter k. For*

$$x + y + z = 1, \quad 2x + ky + z = 2$$

The system is consistent for:
A $k = 1$
B $k = 2$
C $k \neq 2$
D All k

Problem 107 *The system*

$$x + y + z = 2, \quad x + 2y + 3z = 5, \quad 2x + 3y + 4z = 6$$

Is:
A Consistent with unique solution
B Consistent with infinite solutions
C Inconsistent
D Cannot determine

4.10 Try it Yourself

Exercise 51 *(MCQ) The system*

$$x + y + z = 3, \quad 2x + y + 3z = 7, \quad x + 2y + 2z = 6$$

has (A) unique solution
(B) infinitely many solutions
(C) no solution
(D) exactly two solutions

Exercise 52 *(MSQ) For the matrix* $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$*, which statements are true?*
(A) Rank$(A) = 2$
(B) Rank$(A) = 3$
(C) Nullity$(A) = 1$
(D) Nullity$(A) = 0$

Exercise 53 *(Numerical) If the system*

$$x + 2y - z = 1, \quad 2x + 4y - 2z = 2$$

is consistent, find the general solution.

Exercise 54 *(MCQ) For a* 4×5 *system* $Ax = b$ *with Rank*$(A) = 3$*, Rank*$([A|b]) = 3$*, the system has*
(A) unique solution
(B) infinitely many solutions
(C) no solution
(D) exactly 2 solutions

Exercise 55 *(MSQ) For the homogeneous system* $Ax = 0$ *where* A *is* 3×4 *and Rank*$(A) = 2$*, which are true?*
(A) Nullity = 2
(B) Infinitely many solutions
(C) Unique trivial solution only
(D) System is inconsistent

Exercise 56 *(MCQ) Consider*

$$x + y + z = 6, \quad 2x + 3y + z = 10, \quad x + 2y + 2z = k$$

The system has infinitely many solutions for
(A) $k = 9$
(B) $k = 10$
(C) $k = 11$
(D) $k = 12$

Exercise 57 *(Numerical) Solve using Gaussian elimination:*

$$x + 2y + z = 4, \quad 2x + y + 3z = 7, \quad 3x + 4y + 2z = 10$$

Exercise 58 *(MCQ) For a system of equations* $Ax = b$ *with* A 3×3 *and* $\det(A) \neq 0$*, the number of solutions is*
(A) 0
(B) 1
(C) 2
(D) Infinitely many

Exercise 59 *(MSQ) If* A *is* 3×4 *with* $\text{Rank}(A) = 2$*, which of the following hold for* $Ax = 0$*?*
(A) At least one free variable
(B) Nullity = 2
(C) Unique solution
(D) Infinitely many solutions

Exercise 60 *(MCQ) The system*

$$x + y = 3, \quad 2x + 2y = 6, \quad x - y = 1$$

has
(A) unique solution
(B) infinitely many solutions
(C) no solution
(D) exactly two solutions

4.11 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
11	CH 4.1–4.3: Linear Equations (Consistency)	https://youtu.be/vMf94YACk7o	
12	CH 4.4–4.8: Gauss, Jordan, Cramer's & Inverse Methods	https://youtu.be/2VUJ4_TfVgE	
13	CH 4.9: Linear Equations — Solutions to Problems 88–107	https://youtu.be/dUH89T0uxA8	

Chapter 5

Matrix Factorizations

Concept

Matrix Factorization refers to expressing a given matrix as a product of two or more matrices with special properties. Factorization simplifies computations like solving linear systems, computing determinants or inverses, and analyzing structural properties of matrices. LU decomposition is one of the most widely used factorizations for square matrices.

5.1 LU Decomposition

Concept

LU Decomposition factorizes a square matrix A as

$$A = LU,$$

where L is a lower triangular matrix with unit diagonal entries and U is an upper triangular matrix. **Advantages:** 1. Efficiently solves $Ax = b$ for multiple b without repeating elimination. 2. Simplifies determinant computation: $\det(A) = \det(L)\det(U) = \prod_i u_{ii}$. 3. Simplifies computation of A^{-1} by solving $LUx = e_i$ for each column of the identity.

Example 97:

Find the LU decomposition of

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}.$$

Solution: Let

$$L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}.$$

Matching $LU = A$:
First row: $u_{11} = 2$, $u_{12} = 3$. Second row: $l_{21}u_{11} = 4 \implies l_{21} = 2$, $l_{21}u_{12} + u_{22} = 7 \implies 2 \cdot 3 + u_{22} = 7 \implies u_{22} = 1$.
Thus,

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}.$$

Example 98:

Solve

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 7 & 7 \\ -1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 24 \\ 15 \end{bmatrix}$$

using LU decomposition.
Solution: Factor $A = LU$, then solve $Ly = b$ (forward substitution):

$$y_1 = 3, \quad y_2 = 24 - 2 \cdot 3 = 18, \quad y_3 = 15 - (-1 \cdot 3 + 2 \cdot 18) = -21$$

Next, solve $Ux = y$ (backward substitution):

$$x = 2, \quad y = 1, \quad z = 0$$

Example 99:

For

$$A = \begin{bmatrix} 4 & 3 \\ 6 & 3 \end{bmatrix},$$

find LU decomposition.

Solution: Let $L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}$, $U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$.
From $LU = A$: $u_{11} = 4$, $u_{12} = 3$, $l_{21}u_{11} = 6 \implies l_{21} = 1.5$, $l_{21}u_{12} + u_{22} = 3 \implies u_{22} = -1.5$.

$$L = \begin{bmatrix} 1 & 0 \\ 1.5 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & 3 \\ 0 & -1.5 \end{bmatrix}.$$

Example 100:

Determine LU decomposition of

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix}.$$

Solution: Apply elimination:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 1.5 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 0 & -4 \end{bmatrix}.$$

5.2 Applications of LU Factorization

Concept

Applications of LU Factorization:
1. **Solving linear systems efficiently:** Avoid repeated elimination for multiple RHS vectors.
2. **Determinant computation:** $\det(A) = \prod u_{ii}$.
3. **Matrix inversion:** Solve $LUx = e_i$ for each column of identity.
4. **Numerical stability:** LU with pivoting improves solution accuracy.
5. **Engineering and scientific computations:** Widely used in simulations and optimization problems.

5.3 Problems

Problem 108 *(MCQ) Let*

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}.$$

If $A = LU$, where L is lower triangular with unit diagonal, the $(2, 1)$ entry of L is
(A) 1
(B) 2
(C) 0.5
(D) 3

Problem 109 *(MSQ) Let* $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. *Which of the following statements are true?*
(A) LU decomposition exists without pivoting
(B) U will have zero on its diagonal
(C) Determinant of A is 0
(D) System $Ax = b$ may not have unique solution for all b

Problem 110 *(NAT) For*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

compute the determinant using LU decomposition. Provide numerical value.

Problem 111 *(MCQ) Which of the following statements about LU decomposition is true for a square matrix A?*
(A) Always exists without row exchanges
(B) LU decomposition may require pivoting
(C) Determinant of A equals product of diagonal entries of U
(D) L always equals A

Problem 112 *(MSQ) Consider solving $Ax = b$ using LU decomposition. Which statements are true?*
(A) Forward substitution solves $Ly = b$
(B) Backward substitution solves $Ux = y$
(C) Determinant is required to solve system
(D) Multiple RHS vectors require recomputation of L and U

Problem 113 *(MCQ) If A is 3×3 and singular, then in its LU decomposition (if exists without pivoting)*
(A) U has zero on diagonal
(B) L has zero on diagonal
(C) Determinant of U is zero
(D) LU decomposition does not exist

Problem 114 *(MSQ) Let*

$$A = \begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

Identify true statements regarding its LU decomposition:
(A) L is lower triangular with unit diagonal
(B) U is upper triangular
(C) $\det(A) = \prod u_{ii}$
(D) LU decomposition fails because A is singular

Problem 115 *(NAT) Solve $Ax = b$ using LU decomposition for*

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Find x_1.

Problem 116 *(MCQ) LU decomposition is particularly useful in which scenario?*
(A) Solving single 2×2 system once
(B) Solving multiple systems with same A but different b
(C) Computing eigenvalues
(D) Computing inverse by direct formula

Problem 117 *(MSQ) For*

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix},$$

which statements are true?
(A) LU decomposition exists with pivoting
(B) Determinant equals product of diagonal of U
(C) System $Ax = b$ has unique solution for any b
(D) L is not lower triangular

5.4 Try it Yourself

Exercise 61 *(MCQ) Let*

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 4 \\ 3 & 6 & 8 \end{bmatrix}.$$

If $A = LU$, L unit lower triangular, the $(3,1)$ entry of L is
(A) 1
(B) 2
(C) 3
(D) 0

Exercise 62 *(MSQ) Consider*

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 2 & 1 \\ 6 & 1 & 3 \end{bmatrix}.$$

Which statements are true regarding its LU decomposition?
(A) L is unit lower triangular
(B) U is upper triangular
(C) $\det(A) = \prod u_{ii}$
(D) LU decomposition fails

Exercise 63 *(NAT) Solve $Ax = b$ using LU decomposition for*

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

Find x_2.

Exercise 64 *(MCQ) If A is 3×3 and singular, then in its LU decomposition without pivoting:*
(A) U has zero on diagonal
(B) L has zero on diagonal
(C) Determinant of U is zero
(D) LU decomposition does not exist

Exercise 65 *(MSQ) LU decomposition is most advantageous for:*
(A) Solving single linear system once
(B) Solving multiple systems with same A but different b
(C) Determinant and inverse computation
(D) Computing eigenvalues

5.5 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
14	CH 5.1–5.2: Matrix Factorization — LU Decomposition	https://youtu.be/gEzmftGTBaA	
15	CH 5.3: LU Decomposition — Solutions to Problems 108–117	https://youtu.be/Yf2xlZ7byJI	

Chapter 6

Special Matrices

6.1 Projection Matrices

Concept

A **projection matrix** P is a square matrix that satisfies

$$P^2 = P.$$

It "projects" a vector onto a subspace. If v is any vector, Pv is the projection of v onto the subspace. Properties:

1. Eigenvalues of P are 0 or 1.
2. $I - P$ is also a projection matrix.
3. If P is symmetric ($P^T = P$), it is an orthogonal projection.

Example 101:

Let

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Check if P is a projection matrix: $P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = P.$

Example 102:

For

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix},$$

compute $P^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = P$. So P is a projection matrix.

Example 103:

Check if $I - P$ is also a projection:

$$I - P = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}, \quad (I - P)^2 = I - P.$$

Thus, $I - P$ is a projection onto orthogonal subspace.

—

6.2 Orthogonal Matrices

Concept

A square matrix Q is **orthogonal** if

$$Q^T Q = QQ^T = I.$$

Properties:

1. $Q^{-1} = Q^T$.
2. Determinant: $\det(Q) = \pm 1$.
3. Columns (and rows) of Q form an orthonormal set.
4. Eigenvalues satisfy $|\lambda| = 1$.

Example 104:

$$Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Q^T Q = I.$$

So Q is orthogonal.

Example 105:

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad Q^T Q = I.$$

Columns are orthonormal, determinant $= 1$.

Example 106:

Check inverse: $Q^{-1} = Q^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

—

Orthogonal

Concept

A real square matrix A is called **orthogonal** if

$$A^T = A^{-1}, \quad \text{or equivalently } AA^T = A^T A = I.$$

Thus, A must be invertible and its rows and columns form orthonormal sets of vectors.

Example 107:

$$A = \frac{1}{9} \begin{bmatrix} 8 & 4 & 9 \\ 4 & 9 & -9 \\ 9 & -9 & -4 \end{bmatrix}$$

It can be verified that $AA^T = I = A^T A$. Hence, $A^T = A^{-1}$ and A is orthogonal.

Concept

Properties of Orthogonal Matrices:

- The rows of an orthogonal matrix form an orthonormal set.
- The columns of an orthogonal matrix form an orthonormal set.
- If A is orthogonal, then $\det(A) = \pm 1$.

Example 108:

For $n = 2$, every real orthogonal matrix has the form

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Both are orthogonal since $AA^T = I$.

Normal

Concept

A real square matrix A is called **normal** if it commutes with its transpose:

$$AA^T = A^T A.$$

Every symmetric, skew-symmetric, and orthogonal matrix is normal.

Example 109:

$$A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

Then

$$AA^T = \begin{bmatrix} 45 & 0 \\ 0 & 45 \end{bmatrix}, \quad A^T A = \begin{bmatrix} 45 & 0 \\ 0 & 45 \end{bmatrix}.$$

Since $AA^T = A^T A$, the matrix A is normal.

6.3 Idempotent Matrices

Concept

A matrix A is **idempotent** if

$$A^2 = A.$$

Properties:

1. Eigenvalues are 0 or 1.
2. $I - A$ is also idempotent.
3. Rank$(A) =$ tr(A) if A is symmetric.

Example 110:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^2 = A.$$

So A is idempotent.

Example 111:

$$A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad A^2 = A.$$

Hence idempotent.

Example 112:

Check $I - A$:

$$I - A = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}, \quad (I - A)^2 = I - A.$$

—

6.4 Partition (Block) Matrices

Concept

Matrices can be partitioned into submatrices (blocks):

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Block properties:

1. Block multiplication: conformable sizes.
2. Determinant of block-diagonal: $\det(A) = \det(A_{11})\det(A_{22})$.
3. Useful for LU decomposition or large systems.

Example 113:

$$A = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 3 & 4 \end{bmatrix} & \begin{bmatrix} 5 \end{bmatrix} \end{bmatrix}.$$

Matrix is partitioned into 4 blocks.

Example 114:

For block-diagonal

$$A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}, \quad \det(A) = \det(B)\det(C).$$

Example 115:

Block multiplication:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} = \begin{bmatrix} AX + BZ & AY + BW \\ CX + DZ & CY + DW \end{bmatrix}.$$

—

6.5 Properties of Special Matrices

Concept

Summary of key properties:

1. Projection and idempotent: $P^2 = P$, eigenvalues = 0 or 1.
2. Orthogonal: $Q^T Q = I$, inverse = transpose, determinant = ± 1.
3. Block matrices: useful in partitioned operations, determinant of block-diagonal = product of determinants.
4. Symmetric idempotent: rank = trace.

Example 116:

Symmetric idempotent:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{rank}(A) = \text{tr}(A) = 1.$$

Example 117:

Orthogonal check:

$$Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Q^T Q = I, \det(Q) = -1.$$

Example 118:

Projection matrix property:

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad I - P = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}, \quad (I - P)^2 = I - P.$$

—

6.6 Applications of Special Matrices

Concept

Applications:

1. Projection matrices: least squares, computer graphics.
2. Orthogonal matrices: rotations, reflections, numerical stability.
3. Idempotent matrices: statistics (hat matrices), regression analysis.
4. Block matrices: large-scale computation, parallel processing, solving partitioned systems.

6.7 Problems

Problem 118 *(MCQ) Let*

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Which of the following statements is true?
(A) P is idempotent
(B) P is orthogonal
(C) Eigenvalues of P are 0 and 1 only
(D) $\det(P) = 0$

Problem 119 *(MSQ) Identify all orthogonal matrices:*

(A) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(B) $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Problem 120 *(NAT) Eigenvalues of the idempotent matrix*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are: ______.

Problem 121 *(MCQ) Let Q be a 3×3 orthogonal matrix. Then which of the following is always true?*
(A) $Q^{-1} = Q^T$
(B) $\det(Q) = 0$
(C) Q is symmetric
(D) $\|Qx\| = \|x\|$ for any vector x

Problem 122 *(MSQ) Let P be a projection matrix. Which of the following statements are true?*
(A) $P^2 = P$
(B) Eigenvalues of P are 0 or 1 only
(C) P is always invertible
(D) $I - P$ is also a projection matrix

Problem 123 *(MCQ) Consider a block-diagonal matrix*

$$A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}, \quad B, C \text{ are square.}$$

Then which of the following statements is true?
(A) $\det(A) = \det(B) + \det(C)$
(B) $\det(A) = \det(B)\det(C)$

(C) $A^{-1} = \begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix}$ if B and C invertible
(D) Eigenvalues of A are union of eigenvalues of B and C

Problem 124 *(MSQ) Let A be a symmetric idempotent matrix. Which of the following are true?*
(A) $A^2 = A$
(B) $A = A^T$
(C) Rank$(A) =$ tr(A)
(D) Eigenvalues of A can be negative

Problem 125 *(MCQ) Let*

$$P = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then which of the following statements is correct?
(A) P is idempotent
(B) P is a projection matrix
(C) P is orthogonal
(D) rank$(P) = 1$

Problem 126 *(MSQ) Which of the following properties are true for an orthogonal matrix Q?*
(A) $Q^TQ = I$
(B) $Q^{-1} = Q^T$
(C) All eigenvalues satisfy $|\lambda| = 1$
(D) Q must be diagonalizable

Problem 127 *(NAT) Find the rank of the projection matrix*

$$P = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

6.8 Try it Yourself

Exercise 66 *(NAT) Find the determinant of*

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Exercise 67 *(MCQ) Let*

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Which is true?
(A) Q is orthogonal
(B) Q is idempotent
(C) $\det(Q) = 1$
(D) Eigenvalues = 0,1

Exercise 68 *(MSQ) For*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which are true?
(A) Idempotent
(B) Projection
(C) Orthogonal
(D) Rank = 1

Exercise 69 *(NAT) Rank of*

$$Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Exercise 70 *(MCQ) If*

$$P = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & 1.0667 \end{bmatrix}$$

is a projection matrix, its eigenvalues are
(A) 0,1

(B) 1,1
(C) 0,0
(D) 0.5,1.5

6.9 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
16	CH 6.1–6.6: Projection, Orthogonal, Normal, Idempotent	https://youtu.be/GNH-KF8yKxU	
17	CH 6.7: Special Matrices — Solutions to Problems 118–127	https://youtu.be/U3EfHUcj3r4	

Chapter 7

Summary - Very Important

1. Types of Linear Systems

Linear Systems

- **Homogeneous:** $AX = 0$
  - Always consistent
  - Trivial solution: $X = 0$
  - Non-trivial solution if rank$(A) < n$
  - Nullity $= n -$ rank$(A) \to$ number of free variables
- **Non-homogeneous:** $AX = b$
  - Consistent if rank$(A) =$ rank$([A|b])$
  - No solution if rank$(A) \neq$ rank$([A|b])$

Space to write examples: ______________________

2. Rank and Nullity

Rank and Nullity

- Rank: Number of linearly independent rows/columns
- Nullity: $n -$ rank(A)
- Dimension of null space $\to$ solutions to $AX = 0$

- Number of free variables = nullity

3. Rouché–Capelli Theorem (Consistency)

Consistency

System $AX = b$ is consistent iff:

$$\text{rank}(A) = \text{rank}([A|b])$$

- Unique solution: rank = n (full rank)
- Infinite solutions: rank ¡ n and consistent
- No solution: rank([A—b]) ¿ rank(A)

4. Homogeneous Systems

Homogeneous Systems

- Always consistent
- Trivial solution: $X = 0$
- Non-trivial solution exists if rank$(A) < n$
- All solutions are linear combinations of basis vectors of null space

5. Determinants and Invertibility

Determinants

- $\det(A) \neq 0 \Rightarrow$ Full rank, invertible
- $\det(A) = 0 \Rightarrow$ Singular, rank < n
- Non-homogeneous system unique solution only if $\det(A) \neq 0$

6. LU Decomposition

LU Decomposition

$$A = LU, \quad L = \text{lower triangular}, \quad U = \text{upper triangular}$$

- Solve $AX = b$ via:

1. $LY = b$ (forward substitution)
2. $UX = Y$ (backward substitution)

- A invertible $\Rightarrow$ L and U invertible
- Singular $A \Rightarrow$ at least one zero on U diagonal

7. Eigenvalues and Eigenvectors

Eigenvalues

- $Av = \lambda v$
- Trace = sum of eigenvalues
- Determinant = product of eigenvalues
- Rank = number of non-zero eigenvalues
- Symmetric matrices $\rightarrow$ real eigenvalues, orthogonal eigenvectors

8. Underdetermined and Overdetermined Systems

Under/Overdetermined

- $m < n$: fewer equations $\rightarrow$ infinite solutions (if consistent)
- $m > n$: more equations $\rightarrow$ may be inconsistent
- $m = n$: full rank $\rightarrow$ unique solution possible

9. Free Variables and Infinite Solutions

Free Variables

- Free variables $= n -$ rank(A)
- Infinite solutions = linear combination of null space basis
- Homogeneous: all solutions = linear combinations of null space vectors

10. Parameter-Dependent Systems

Parameters

- Compute determinant/rank symbolically
- Solve for parameters:
  - Infinitely many solutions $\rightarrow$ rank drops
  - Unique solution $\rightarrow$ full rank
  - No solution $\rightarrow$ augmented rank $>$ rank

11. Linear Dependence of Columns

Linear Dependence

- Columns $a_1, \ldots, a_n$ linearly dependent if $\sum c_i a_i = 0, c_i \neq 0$
- Implication: solution may be infinite or none depending on augmented rank

12. Key Tricks and Observations

Tricks

- Sum of singular matrices $\rightarrow$ may be non-singular
- Sum of non-singular matrices $\rightarrow$ may be singular
- Zero determinant $\rightarrow$ each row/column $=$ linear combination of others
- Eigenvalues of symmetric $\rightarrow$ real
- LU decomposition: forward/backward substitution

Summary

1. Types of Systems

Homogeneous, Non-homogeneous; Trivial/non-trivial solutions

2. Rank and Nullity

Rank $=$ independent rows/columns; Nullity $=$ n - rank

3. Consistency (Rouché–Capelli)

System consistent if rank(A) = rank([A—b])

4. Homogeneous Systems

Always consistent; non-trivial if rank ¡ n

5. Determinants and Invertibility

det(A) 0 → invertible; det(A) = 0 → singular

6. LU Decomposition

Solve AX=b via LY=b, UX=Y; L,U invertible if A invertible

7. Eigenvalues and Eigenvectors

$Av = \lambda v$; Symmetric → real eigenvalues, orthogonal eigenvectors

8. Under/Overdetermined Systems

m<n → infinite solutions; m¿n → may be inconsistent

9. Free Variables and Infinite Solutions

Free variables = n - rank(A); infinite solutions = linear combinations

10. Parameter-Dependent Systems

Use determinant/rank symbolically to find parameter values for solution type

11. Linear Dependence of Columns

Columns dependent → solution may be infinite or none depending on augmented rank

7.1 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
Bonus	Summary: Col Space, Independence, Span, Basis, Rank, Null Space	https://youtu.be/ltI1tXG8yUE	

Chapter 8

GATE PYQs

8.1 Problems

GATEPYQ 1 *Consider a matrix* $A = uv^T$ *where* $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. *Note that* v^T *denotes the transpose of* v. *The largest eigenvalue of* A *is .*

GATEPYQ 2 *If* A *is a skew-symmetric matrix, then* A^T *is:*

(A) Diagonal matrix
(B) A
(C) 0
(D) $-A$

GATEPYQ 3 *If the characteristic polynomial of a* 3×3 *matrix* M *over* $\mathbb{R}$ *is* $\lambda^3 - 4\lambda^2 + a\lambda + 30$, $a \in \mathbb{R}$, *and one eigenvalue of* M *is 2, then the largest among the absolute values of the eigenvalues of* M *is .*

GATEPYQ 4 *Let* $P = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ *and* $Q = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$ *be two matrices. Then the rank of* $P + Q$ *is .*

GATEPYQ 5 *Consider a system of linear equations* $PX = Q$ *where* $P \in \mathbb{R}^{3\times3}$ *and* $Q \in \mathbb{R}^{3\times1}$. *Suppose* P *has an* LU *decomposition,* $P = LU$, *where*

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$ *and*

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

Which of the following statement(s) is/are TRUE?

(A) The system $PX = Q$ can be solved by first solving $LY = Q$ and then $UX = Y$.
(B) If P is invertible, then both L and U are invertible.
(C) If P is singular, then at least one of the diagonal elements of U is zero.
(D) If P is symmetric, then both L and U are symmetric.

GATEPYQ 6 *Let $L, M,$ and N be non-singular matrices of order 3 satisfying the equations $L^2 = L^{-1}$, $M = L^8$, $N = L^2$.*
Which ONE of the following is the value of the determinant of $(M - N)$?

(A) 0
(B) 1
(C) 2
(D) 3

GATEPYQ 7 *Let* $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$

Then which ONE of the following is A^8?

(A) $\begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$

(B) $\begin{bmatrix} 125 & 0 \\ 0 & 125 \end{bmatrix}$

(C) $\begin{bmatrix} 625 & 0 \\ 0 & 625 \end{bmatrix}$

(D) $\begin{bmatrix} 3125 & 0 \\ 0 & 3125 \end{bmatrix}$

GATEPYQ 8 *Let A be a 2×2 matrix given by* $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

What are the eigenvalues of the matrix A^{13}?

(A) 1, -1
(B) 2, -2
(C) 4, -4
(D) 64, -64

GATEPYQ 9 *Let A be an $n \times n$ matrix over the set of all real numbers $\mathbb{R}$.*
Let B be a matrix obtained from A by swapping two rows.
Which of the following statements is/are TRUE?

(A) The determinant of B is the negative of the determinant of A.
(B) If A is invertible, then B is also invertible.
(C) If A is symmetric, then B is also symmetric.
(D) If the trace of A is zero, then the trace of B is also zero.

GATEPYQ 10 *Let A be any $n \times m$ matrix, where $m > n$.*
Which of the following statements is/are TRUE about the system of linear equations $Ax = 0$?

(A) There exist at least $m - n$ linearly independent solutions to this system
(B) There exist $m - n$ linearly independent vectors such that every solution is a linear combination of these vectors
(C) There exists a non-zero solution in which at least $m - n$ variables are 0
(D) There exists a solution in which at least n variables are non-zero

GATEPYQ 11 *The product of all eigenvalues of the matrix*

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$ *is*

(A) -1
(B) 0
(C) 1
(D) 2

GATEPYQ 12 *Let A be the adjacency matrix of the graph with vertices $\{1, 2, 3, 4, 5\}$.*
Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ be the five eigenvalues of A.
The value of $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 =$

(A) 0.0
(B) 1.0
(C) 5.0
(D) Cannot be determined

GATEPYQ 13 *Let A be an $n \times n$ real-valued square symmetric matrix of rank 2 with $\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}^2 = 50$. Consider the following statements:*
(I) One eigenvalue must be in $[-5, 5]$
(II) The eigenvalue with the largest magnitude must be strictly greater than 5.
Which of the above statements about eigenvalues of A is/are necessarily CORRECT?

(A) Both (I) and (II)
(B) (I) only
(C) (II) only
(D) Neither (I) nor (II)

GATEPYQ 14 *Let $c_1, \ldots, c_n$ be scalars, not all zero, such that $\sum_{i=1}^{n} c_i a_i = 0$ where a_i are column vectors in $\mathbb{R}^n$. Consider the set of linear equations $Ax = b$ where $A = [a_1 \ldots a_n]$ and $b = \sum_{i=1}^{n} a_i$. The set of equations has:*

(A) a unique solution at $x = J_n$, where J_n denotes an n-dimensional vector of all 1's
(B) no solution
(C) infinitely many solutions
(D) finitely many solutions

GATEPYQ 15 *Suppose that the eigenvalues of matrix A are 1, 2, 4. The determinant of $(A-1)^T$ is .*

GATEPYQ 16 *Consider the systems, each consisting of m linear equations in n variables:*

I. If $m \ll n$, then all such systems have a solution
II. If $m \gg n$, then none of these systems has a solution
III. If $m = n$, then there exists a system which has a solution
Which one of the following is CORRECT?

(A) I, II and III are true
(B) Only II and III are true
(C) Only III is true
(D) None of them is true

GATEPYQ 17 *Two eigenvalues of a 3×3 real matrix P are $(2 \pm i)$ and 3. The determinant of P is .*

GATEPYQ 18 *If the following system has a non-trivial solution:*
$px + qy + rz = 0$
$qx + ry + pz = 0$
$rx + py + qz = 0$
Then which one of the following options is TRUE?

(A) $p - q + r = 0$ or $p = q = -r$
(B) $p + q - r = 0$ or $p = -q = r$
(C) $p + q + r = 0$ or $p = q = r$
(D) $p - q + r = 0$ or $p = -q = -r$

GATEPYQ 19 *In the given matrix* $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$, *one of the eigenvalues is 1. The eigenvectors corresponding to the eigenvalue 1 are:*

(A) $\{\alpha(4, 2, 1) \mid \alpha \neq 0, \alpha \in \mathbb{R}\}$
(B) $\{\alpha(-4, 2, 1) \mid \alpha \neq 0, \alpha \in \mathbb{R}\}$
(C) $\{\alpha(2, 0, 1) \mid \alpha \neq 0, \alpha \in \mathbb{R}\}$
(D) $\{\alpha(-2, 0, 1) \mid \alpha \neq 0, \alpha \in \mathbb{R}\}$

GATEPYQ 20 *Perform the following operations on the matrix* $\begin{bmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{bmatrix}$:

(i) Add the third row to the second row
(ii) Subtract the third column from the first column
The determinant of the resultant matrix is .

GATEPYQ 21 *Consider the 2x2 matrix* $A = \begin{bmatrix} 1 & b \\ 4 & a \end{bmatrix}$ *where the eigenvalues are -1 and 7. The values of* a *and* b *are:*

(A) $a = 6, b = 4$
(B) $a = 4, b = 6$
(C) $a = 3, b = 5$
(D) $a = 5, b = 3$

GATEPYQ 22 *In the LU decomposition of the matrix* $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$*, if the diagonal elements of* U *are both 1, then the lower diagonal entry* l_{22} *of* L *is .*

GATEPYQ 23 *The rank of the matrix* $A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{bmatrix}$ *is:*

(A) 0
(B) 1
(C) 2
(D) 3

GATEPYQ 24 *Let* A *be a square matrix of size* $n \times n$*. Consider the following pseudocode:*

```
C = 100; for i = 1 to n do for j = 1 to n do { Temp = A[i][j] + C; A[i][j] = A[j][i];
A[j][i] = Temp - C; } for i = 1 to n do for j = 1 to n do output(A[i][j]);
```

The expected output is:

(A) The matrix A *itself*
(B) Transpose of the matrix A
(C) Adding 100 to the upper diagonal elements and subtracting 100 from lower diagonal elements of A
(D) None of these

GATEPYQ 25 *If* V_1 *and* V_2 *are 4-dimensional subspaces of a 6-dimensional vector space* V*, then the smallest possible dimension of* $V_1 \cap V_2$ *is .*

GATEPYQ 26 *Which one of the following statements is TRUE about every* $n \times n$ *matrix with only real eigenvalues?*

(A) If the trace of the matrix is positive and the determinant of the matrix is negative, at least one of its eigenvalues is negative
(B) If the trace of the matrix is positive, all its eigenvalues are positive
(C) If the determinant of the matrix is positive, all its eigenvalues are positive
(D) If the product of the trace and determinant of the matrix is positive, all its eigenvalues are positive

GATEPYQ 27 *The product of the non-zero eigenvalues of the matrix* $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ *is .*

GATEPYQ 28 *The value of the dot product of the eigenvectors corresponding to any pair of different eigenvalues of a 4-by-4 symmetric positive definite matrix is*

GATEPYQ 29 *Consider the following system of equations:*
$3x + 2y = 1$
$4x + 7z = 1$
$x + y + z = 3$
$x - 2y + 7z = 0$
The number of solutions for this system is .

GATEPYQ 30 *What is the matrix transformation which takes the independent vectors* $(1, 2)$ *and* $(2, 5)$ *and transforms them to* $(1, 1)$ *and* $(3, 2)$ *respectively?*

(A) $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 0 & 0 \\ 0.5 & 0.5 \end{bmatrix}$

(C) $\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

GATEPYQ 31 *Which one of the following does NOT equal* $\begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix}$?

(A) $\begin{bmatrix} 1 & x(x+1) & x+1 \\ 1 & y(y+1) & y+1 \\ 1 & z(z+1) & z+1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & (x+1) & x^2+1 \\ 1 & (y+1) & y^2+1 \\ 1 & (z+1) & z^2+1 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{bmatrix}$

(D) $\begin{bmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{bmatrix}$

GATEPYQ 32 *Let A be the 2×2 matrix with elements $a_{11} = a_{12} = a_{21} = +1$ and $a_{22} = -1$. Then the eigenvalues of A^{19} are:*

(A) 1024 and -1024
(B) $1024/2$ *and* $-1024/2$
(C) $4/2$ *and* $-4/2$
(D) $512/2$ *and* $-512/2$

GATEPYQ 33 *Let* $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}$ *and*

$B = \begin{bmatrix} 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$.

Let $\det(A)$ *and* $\det(B)$ *denote the determinants of the matrices* A *and* B, *respectively. Which one of the options given below is TRUE?*

(A) $\det(A) = \det(B)$
(B) $\det(B) = -\det(A)$
(C) $\det(A) = 0$
(D) $\det(AB) = \det(A) + \det(B)$

GATEPYQ 34 *Which of the following is/are the eigenvector(s) for the matrix*
$\begin{bmatrix} -9 & -6 & -2 & -4 \\ -8 & -6 & -3 & -1 \\ 20 & 15 & 8 & 5 \\ 32 & 21 & 7 & 12 \end{bmatrix}$?

(A) $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

(C) $\begin{bmatrix} -1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$

(D) $\begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \end{bmatrix}$

GATEPYQ 35 *Consider solving the system of simultaneous equations using LU decomposition:*
$x_1 + x_2 - 2x_3 = 4$
$x_1 + 3x_2 - x_3 = 7$
$2x_1 + x_2 - 5x_3 = 7$

Where L and U are denoted as:

$$L = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix}, U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

Which one of the following is the correct combination of values for L_{32}, U_{33}, x_1?

(A) $L_{32} = 2, U_{33} = -\frac{1}{2}, x_1 = -1$
(B) $L_{32} = 2, U_{33} = 2, x_1 = -1$
(C) $L_{32} = -\frac{1}{2}, U_{33} = 2, x_1 = 0$
(D) $L_{32} = -\frac{1}{2}, U_{33} = -\frac{1}{2}, x_1 = 0$

GATEPYQ 36 *Consider the following statements with respect to the matrices* $A_{m\times n}, B_{n\times m}, C_{n\times n}, D_{n\times n}$*:*

Statement 1: $\mathrm{tr}(AB) = \mathrm{tr}(BA)$
Statement 2: $\mathrm{tr}(CD) = \mathrm{tr}(DC)$

Which one of the following holds?

(A) Statement 1 is correct and Statement 2 is wrong.
(B) Statement 1 is wrong and Statement 2 is correct.
(C) Both Statement 1 and Statement 2 are correct.
(D) Both Statement 1 and Statement 2 are wrong.

GATEPYQ 37 *Suppose* P *is a* 4×5 *matrix such that every solution of the equation* $Px = 0$ *is a scalar multiple of*

$$\begin{bmatrix} 2 \\ 5 \\ 4 \\ 3 \\ 1 \end{bmatrix}.$$

The rank of P *is*

GATEPYQ 38 *Consider the following matrix:*

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

The largest eigenvalue of the above matrix is

GATEPYQ 39 *Let* A *and* B *be two* $n \times n$ *matrices over real numbers. Let* $\mathrm{rank}(M)$ *and* $\det(M)$ *denote the rank and determinant of a matrix* M*, respectively. Consider the following statements:*

I. $\mathrm{rank}(AB) = \mathrm{rank}(A)\,\mathrm{rank}(B)$
II. $\det(AB) = \det(A)\det(B)$
III. $\mathrm{rank}(A + B) \leq \mathrm{rank}(A) + \mathrm{rank}(B)$

IV. $\det(A+B) \leq \det(A) + \det(B)$

Which of the above statements are TRUE?

(A) I and II only
(B) I and IV only
(C) II and III only
(D) III and IV only

GATEPYQ 40 *Consider the following matrix:*

$$R = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \end{bmatrix}$$

The absolute value of the product of eigenvalues of R is

GATEPYQ 41 *Let X be a square matrix. Consider the following two statements:*

I. X is invertible
II. Determinant of X is non-zero

Which one of the following is TRUE?

(A) I implies II; II does not imply I
(B) II implies I; I does not imply II
(C) I does not imply II; II does not imply I
(D) I and II are equivalent statements

GATEPYQ 42 *Consider a matrix P whose only eigenvectors are the multiples of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Consider the following statements:*

(I) P does not have an inverse
(II) P has a repeated eigenvalue
(III) P cannot be diagonalized

Which one of the following options is correct?

(A) Only I and III are necessarily true
(B) Only II is necessarily true
(C) Only I and II are necessarily true
(D) Only II and III are necessarily true

GATEPYQ 43 *What is the matrix that represents rotation of an object by θ_0 about the origin in 2D?*

(A) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(B) $\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

(C) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

(D) $\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

GATEPYQ 44 *If A and B are square matrices of the same order and A is symmetric, then $B^T AB$ is:*

(A) Skew symmetric
(B) Symmetric
(C) Orthogonal
(D) Idempotent

GATEPYQ 45 *Consider the matrix* $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 3 \end{bmatrix}$. *Which of the following provides the correct values of eigenvalues?*

(A) 1, 4, 3
(B) 3, 7, 3
(C) 7, 3, 2
(D) 1, 2, 3

GATEPYQ 46 *Consider the matrix* $A = \begin{bmatrix} 2 & x \\ 3 & y \end{bmatrix}$. *If the eigenvalues of* A *are 4 and 8, then:*

(A) $x = 4, y = 10$
(B) $x = 5, y = 8$
(C) $x = -3, y = 9$
(D) $x = -4, y = 10$

GATEPYQ 47 *If* M *is a square matrix with zero determinant, which of the following assertions are correct?*
S1: Each row of M *can be represented as a linear combination of the other rows*
S2: Each column of M *can be represented as a linear combination of the other columns*
S3: $MX = 0$ *has a nontrivial solution*
S4: M *has an inverse*
(A) S3 and S2
(B) S1 and S4
(C) S1 and S3
(D) S1, S2 and S3

GATEPYQ 48 *How many of the following matrices have an eigenvalue 1?*

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

(A) one
(B) two
(C) three
(D) four

GATEPYQ 49 *Let* $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$. *What is the maximum value of* $x^T Ax$ *where the maximum is taken over all unit eigenvectors* x*?*

(A) 5
(B) $\frac{5+\sqrt{5}}{2}$
(C) 3
(D) $\frac{5-\sqrt{5}}{2}$

GATEPYQ 50 *What are the eigenvalues of the matrix* $P = \begin{bmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{bmatrix}$*?*

(A) $a, a-2, a+2$
(B) a, a, a
(C) 0, a, 2a
(D) -a, 2a, 2a

GATEPYQ 51 F *is an* $n \times n$ *real matrix,* b *is an* $n \times 1$ *real vector. Suppose there are two vectors* $u \neq v$ *such that* $Fu = b$ *and* $Fv = b$*. Which statement is FALSE?*

(A) Determinant of F *is zero*
(B) There are an infinite number of solutions to $Fx = b$
(C) There is an $x \neq 0$ *such that* $Fx = 0$
(D) F *must have two identical rows*

GATEPYQ 52 *The determinant of the matrix* $\begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix}$ *is:*

(A) -1
(B) 0
(C) 1
(D) 2

GATEPYQ 53 *Consider the system of equations in three real variables* x_1, x_2, x_3*:*

$$2x_1 - x_2 + 3x_3 = 1$$
$$3x_1 + 2x_2 + 5x_3 = 2$$
$$-x_1 + 4x_2 + x_3 = 3$$

The system of equations has:

(A) no solution
(B) a unique solution
(C) more than one but a finite number of solutions

(D) an infinite number of solutions

GATEPYQ 54 *If matrix* $X = \begin{bmatrix} a & -a^2+a-1 \\ 1 & 1-a \end{bmatrix}$ *and* $X^2 - X + I = O$*, where* I *is the identity and* O *is the zero matrix, then the inverse of* X *is:*

(A) $\begin{bmatrix} 1-a & -1 \\ a^2 & a \end{bmatrix}$

(B) $\begin{bmatrix} 1-a & -1 \\ a^2-a+1 & a \end{bmatrix}$

(C) $\begin{bmatrix} -a & -a^2+a-1 \\ 1 & 1-a \end{bmatrix}$

(D) $\begin{bmatrix} a^2-a+1 & a \\ 1 & 1-a \end{bmatrix}$

GATEPYQ 55 *How many solutions does the system:* $-x + 5y = -1$, $x - y = 2$, $x + 3y = 3$ *have?*

(A) infinitely many
(B) two distinct solutions
(C) unique
(D) none

GATEPYQ 56 *Let* A, B, C, D *be* $n \times n$ *matrices, each with non-zero determinant. If* $ABCD = I$*, then* B^{-1} *is:*

(A) $A^{-1}D^{-1}C^{-1}$
(B) CDA
(C) ADC
(D) Does not necessarily exist

GATEPYQ 57 *The number of different* $n \times n$ *symmetric matrices with each element being either 0 or 1 is:*

(A) 2^n
(B) 2^{n^2}
(C) $2^{(n^2+n)/2}$
(D) $2^{(n^2-n)/2}$

GATEPYQ 58 *Consider the system:* $\begin{bmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 5 \\ 7 \end{bmatrix}$. *For how many values of* α *does this system have infinitely many solutions?*

(A) 0
(B) 1
(C) 2
(D) infinitely many

GATEPYQ 59 *The rank of the matrix* $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ *is:*

(A) 4
(B) 2
(C) 1
(D) 0

GATEPYQ 60 *The determinant of the matrix* $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 8 & 1 & 7 & 2 \\ 2 & 0 & 2 & 0 \\ 9 & 0 & 6 & 1 \end{bmatrix}$ *is:*

(A) 4
(B) 0
(C) 15
(D) 20

GATEPYQ 61 *The rank of the matrix* $\begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 4 & 2 & 3 & 1 \\ 3 & 12 & 24 & 21 \end{bmatrix}$ *is:*

(A) 3
(B) 1
(C) 2
(D) 4

GATEPYQ 62 *Consider the set of equations:* $x + 2y = 5$, $4x + 8y = 12$, $3x + 6y + 3z = 15$. *This set:*

(A) has unique solution
(B) has no solution
(C) has finite number of solutions
(D) has infinite number of solutions

GATEPYQ 63 *Let* $A = (a_{ij})$ *be an* n*-rowed square matrix and* I_{12} *be the matrix obtained by interchanging the first and second rows of the identity. Then* AI_{12} *is such that its:*

(A) first row is the same as its second row
(B) first row is the same as the second row of A
(C) first column is the same as the second column of A
(D) first row is all zero

GATEPYQ 64 *The determinant of the matrix* $\begin{bmatrix} 6 & -8 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ *is:*

(A) 11
(B) -48
(C) 0
(D) -24

GATEPYQ 65 *Let $AX = b$ be a system with A an $m \times n$ matrix, b an $m \times 1$ vector, and X an $n \times 1$ vector of unknowns. Which is FALSE?*

(A) The system has a solution iff A and augmented matrix $[A|b]$ have the same rank
(B) If $m < n$ and $b = 0$, the system has infinitely many solutions
(C) If $m = n$ and $b \neq 0$, the system has a unique solution
(D) The system will have only trivial solution when $m = n$, $b = 0$ and rank$(A) = n$

GATEPYQ 66 *The rank of the $(n+1) \times (n+1)$ matrix* $\begin{bmatrix} 1 & 1 & \dots & 1 \\ a & a & \dots & a \\ a^2 & a^2 & \dots & a^2 \\ \vdots & \vdots & & \vdots \\ a^n & a^n & \dots & a^n \end{bmatrix}$ *is:*

(A) 1
(B) 2
(C) n
(D) Depends on the value of a

GATEPYQ 67 *The rank of the matrix* $\begin{bmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix}$ *is:*

(A) 0
(B) 1
(C) 2
(D) 3

GATEPYQ 68 *Let A and B be real symmetric matrices of size $n \times n$. Then which is true?*

(A) $AA' = I$
(B) $A = A^{-1}$
(C) $AB = BA$
(D) $(AB)' = BA$

GATEPYQ 69 *The eigenvector(s) of the matrix* $\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\alpha \neq 0$, *is(are):*

(A) $(0, 0, \alpha)$
(B) $(\alpha, 0, 0)$
(C) $(0, 0, 1)$
(D) $(0, \alpha, 0)$

GATEPYQ 70 *Consider the following statements:*

S1: The sum of two singular $n \times n$ *matrices may be non-singular. S2: The sum of two* $n \times n$ *non-singular matrices may be singular.*

Which is correct?

(A) S1 and S2 are both true
(B) S1 true, S2 false
(C) S1 false, S2 true
(D) S1 and S2 are both false

8.2 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
18	CH 8: GATE PYQ Solutions 1–35	https://youtu.be/RciMJ0jm_zQ	
19	CH 8: GATE PYQ Solutions 36–70	https://youtu.be/felSe7CIo0g	
20	CH 8: GATE PYQs 10, 14, 16, 47, 51, 65	https://youtu.be/qSzSBX_dD1o	

Chapter 9

Solutions

Problems Covered	YouTube Link	QR Code
Solutions to Problems 1–18	https://youtu.be/JYNda2xGvZw	
Solutions to Problems 19–31	https://youtu.be/WboSez4cRIU	
Rank & Nullity – Solutions to Problems 31–60	https://youtu.be/hVI2PC1fTeM	

Eigen Vectors – Solutions to Problems 61–87	https://youtu.be/UYBow0cStbQ	
Linear Equations – Solutions to Problems 88–107	https://youtu.be/dUH89T0uxA8	
LU Decomposition – Solutions to Problems 108–117	https://youtu.be/Yf2xlZ7byJI	
Special Matrices – Solutions to Problems 118–127	https://youtu.be/U3EfHUcj3r4	
GATE PYQ Solutions 1–35	https://youtu.be/RciMJ0jm_zQ	

GATE PYQ Solutions 36–70	https://youtu.be/felSe7CIo0g	
GATE PYQs 10, 14, 16, 47, 51, 65	https://youtu.be/qSzSBX_dD1o	

Bibliography

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