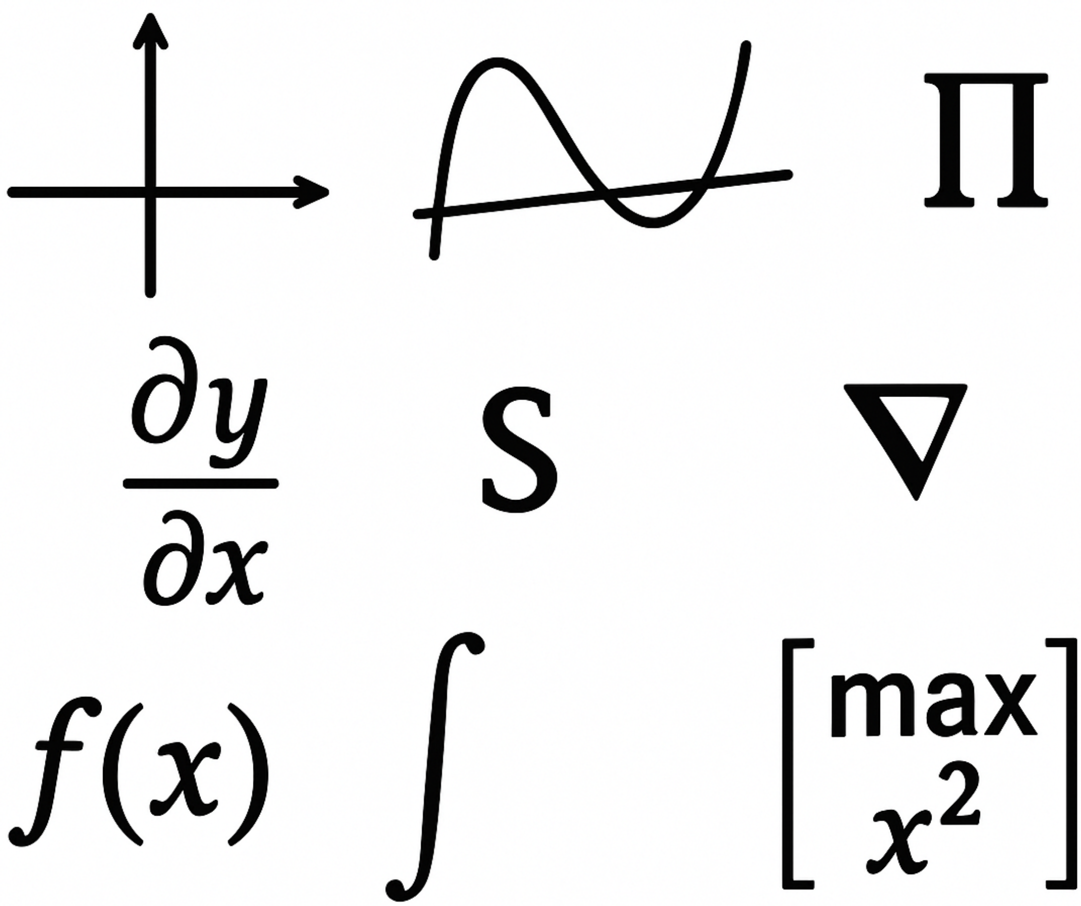
GATE – Data Science and Artificial Intelligence (DA)

Calculus and Optimization



GateXAIML

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Contents

About the Book

Artificial Intelligence and Machine Learning (AI/ML) are transforming industries across the globe — from healthcare and finance to transportation and education. From medical diagnosis systems and fraud detection to personalized recommendations and autonomous vehicles, AI/ML is shaping the way we live, work, and interact with technology.

To support this rapidly growing field, the GATE Data Science and Artificial Intelligence (DA) exam was introduced as a national-level gateway to higher studies, research, and employment opportunities in top institutions and organizations. The exam tests a candidate's proficiency in mathematics, programming, data handling, machine learning, and AI fundamentals.

This book is a compact and comprehensive guide for GATE DA aspirants. It is designed to help learners build a strong conceptual foundation while developing the problem-solving skills required for the exam. Many solved examples are included to illustrate key concepts, and each chapter features carefully crafted problems for practice.

Solutions to selected problems and topic-wise lectures will be discussed in detail on my YouTube channel (@GATEXAiml). All the concepts covered in the book will also be taught step-by-step through video tutorials, making this a complete learning resource for GATE DA preparation.

This book is designed for aspirants of the GATE DA exam focusing on **Calculus and Optimization**. It systematically covers theory, solved examples, and practice problems aligned with the official syllabus.

Dedicated to all my Gurus and Students.
"Knowledge grows only when shared — and it must remain free, for that is how it thrives."

Calculus and Optimization - Syllabus

Functions of a single variable, limit, continuity and differentiability, Taylor series, maxima and minima, optimization involving a single variable.

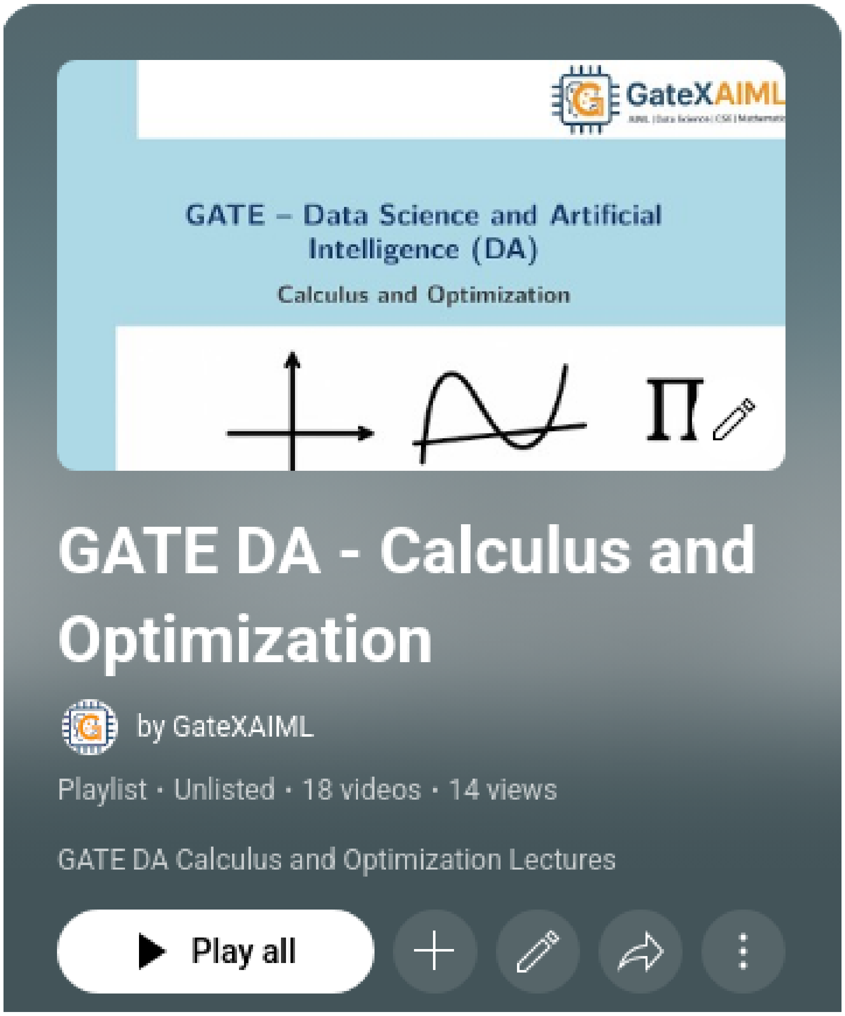
STOP!

Attention!

Some examples solved in video lectures are different from those given in this book.

The procedure to solve problems and examples is well explained in the video lectures, and it is highly recommended to go through the video lectures for complete understanding.

Official Video Playlist



Watch on YouTube

Chapter 1

Functions

1.1 Introduction

Given two sets A and B, a **relation** from A to B is a set of ordered pairs (x, y), where $x \in A$ and $y \in B$. A relation from A to B defines a relationship between these two sets.

A **function** is a special type of relation in which each element of the first set is related to **exactly one** element of the second set. The element of the first set is called the **input**, and the element of the second set is called the **output**. Functions are used extensively in mathematics to describe relationships between sets.

For any function, when we know the input, the output is determined, so we say that the output is a **function of the input**. For example:

- The area of a square is determined by its side length. Area (output) is a function of side length (input).
- The velocity of a ball thrown in the air can be described as a function of the amount of time the ball is in the air.
- The cost of mailing a package is a function of the weight of the package.

Definition

A **function** f consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The set of inputs is called the **domain** of the function. The set of outputs is called the **range** of the function.

Example 1: Squaring function

Consider the function f where the domain is the set of all real numbers and the rule is to square the input. Then, the input $x = 3$ is assigned to the output $3^2 = 9$.
Since every nonnegative real number has a real square root, every nonnegative number is in the

range of this function. Negative numbers are not in the range. Hence, the range is the set of nonnegative real numbers.

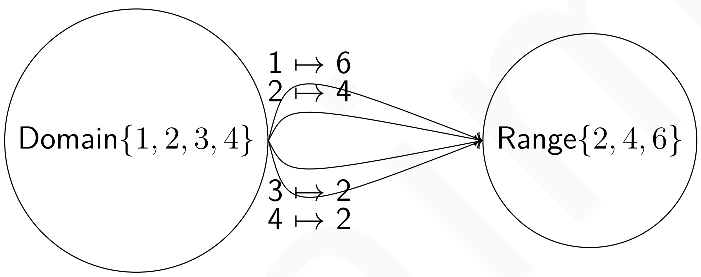
For a general function f with domain D, we often denote:

$$x = \text{input (independent variable)}, \quad y = \text{output (dependent variable)}, \quad y = f(x)$$

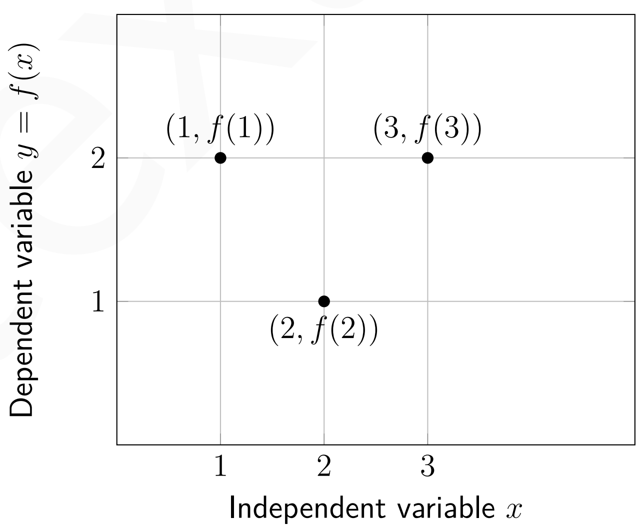
For example, the squaring function can be written as $f(x) = x^2$.

Visualizing a Function

A function can be visualized as an input/output device. A function maps every element



in the domain to exactly one element in the range. Two inputs may map to the same output. Graph of a function f with domain $\{1, 2, 3\}$ and range $\{1, 2\}$.



Graph of a Function

A function can also be visualized by plotting points (x, y) where $y = f(x)$ in the coordinate plane.

Example 2:

Consider $f(x) = 3 - x$ with domain $D = \{1, 2, 3\}$. The graph consists of points $(x, f(x))$:

$$(1, 2), (2, 1), (3, 0)$$

Domain and Range

Every function has a domain. If a function is given by an equation with no specified domain, the domain is taken to be all real numbers for which $f(x)$ is real.

- For $f(x) = x^2$, the domain is all real numbers.
- For $f(x) = \sqrt{x}$, the domain is all nonnegative real numbers.

We can describe infinite sets using **set-builder** or **interval notation**:

$$\{x \mid 1 < x < 5\} = (1, 5)$$

$$[1, 5] = \{x \mid 1 \leq x \leq 5\}$$

$$[0, \infty) = \{x \mid 0 \leq x\}$$

$$(-\infty, 0] = \{x \mid x \leq 0\}$$

$$(-\infty, \infty) = \{x \mid x \text{ is any real number}\}$$

Piecewise-Defined Functions

Some functions are defined by different rules for different parts of the domain. These are called piecewise-defined functions:

$$f(x) = \begin{cases} 3x + 1, & x \geq 2 \\ x^2, & x < 2 \end{cases}$$

Example 3:

Evaluate $f(x)$ for $x = 5$ and $x = -1$:

$$f(5) = 3(5) + 1 = 16, \quad f(-1) = (-1)^2 = 1$$

Open and Closed Intervals

Open and Closed Intervals

In mathematics, an **interval** is a set of real numbers lying between two endpoints. Intervals can be classified as:

- **Open interval** (a, b): Includes all numbers strictly between a and b, but **does not include** the endpoints a and b. Example: $(1, 5) = \{x \mid 1 < x < 5\}$.
- **Closed interval** $[a, b]$: Includes all numbers between a and b, **including** the endpoints a and b. Example: $[1, 5] = \{x \mid 1 \leq x \leq 5\}$.
- **Half-open (or half-closed) intervals**:
  - $[a, b)$ includes a but not b.
  - $(a, b]$ includes b but not a.

1.2 Classes of Functions

Functions can be categorized into several types depending on their properties and rules of assignment. Understanding these classes helps in analyzing functions effectively.

Concept: Classes of Functions

Some common classes of functions include:

1. Linear Functions

A **linear function** has the form:

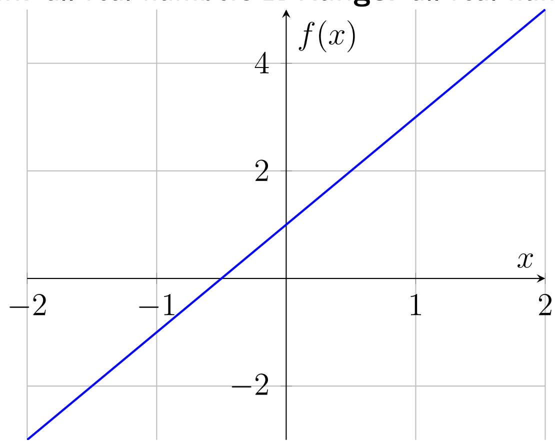
$$f(x) = mx + b$$

where m and b are real constants. The graph of a linear function is a straight line.

Example 4:

Consider $f(x) = 2x + 1$.

Domain: all real numbers $\mathbb{R}$ **Range:** all real numbers $\mathbb{R}$



2. Quadratic Functions

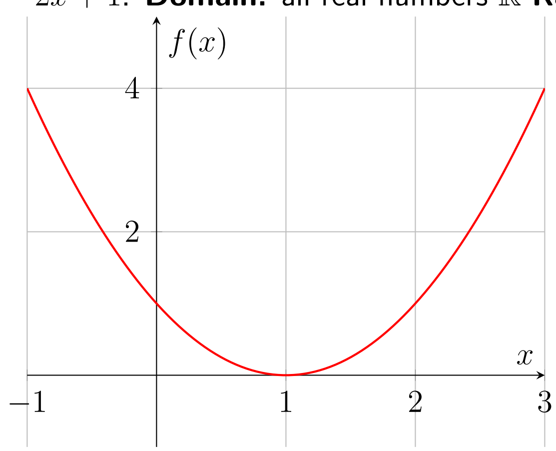
A **quadratic function** has the form:

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

Its graph is a parabola.

Example 5:

Consider $f(x) = x^2 - 2x + 1$. **Domain:** all real numbers $\mathbb{R}$ **Range:** $[0, \infty)$



3. Polynomial Functions

A **polynomial function** is of the form:

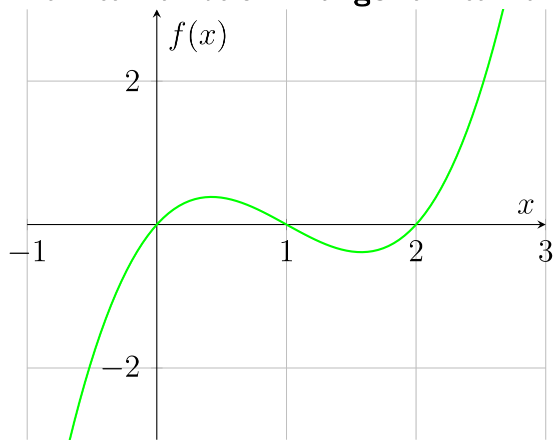
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where a_i are constants and n is a non-negative integer.

Example 6:

Consider $f(x) = x^3 - 3x^2 + 2x$.

Domain: all real numbers $\mathbb{R}$ **Range:** all real numbers $\mathbb{R}$



4. Rational Functions

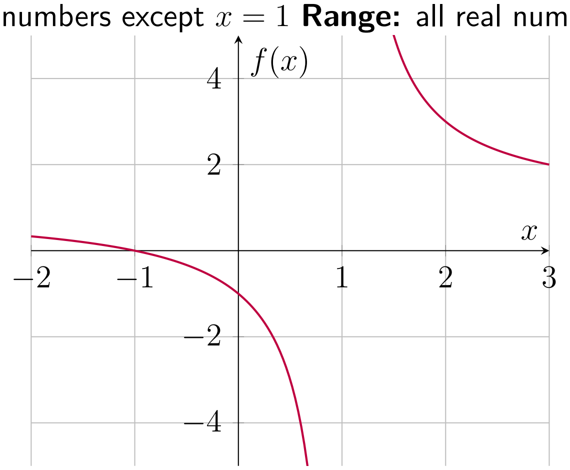
A **rational function** is the ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}, \quad Q(x) \neq 0$$

Example 7:

Consider $f(x) = \frac{x+1}{x-1}$.

Domain: all real numbers except $x = 1$ **Range:** all real numbers except $y = 1$



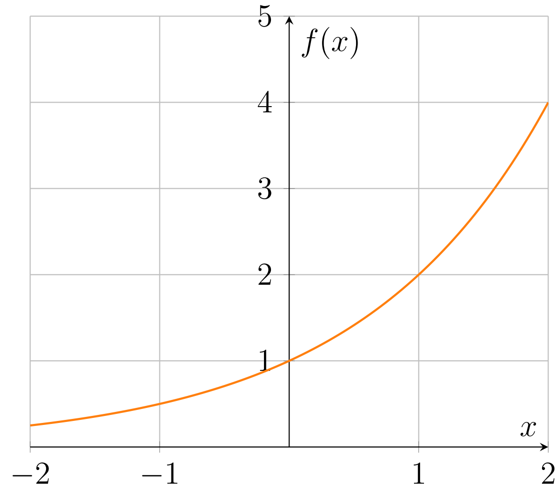
5. Exponential Functions

An **exponential function** has the form:

$$f(x) = a^x, \quad a > 0, a \neq 1$$

Example 8:

Consider $f(x) = 2^x$. **Domain:** all real numbers $\mathbb{R}$ **Range:** $(0, \infty)$



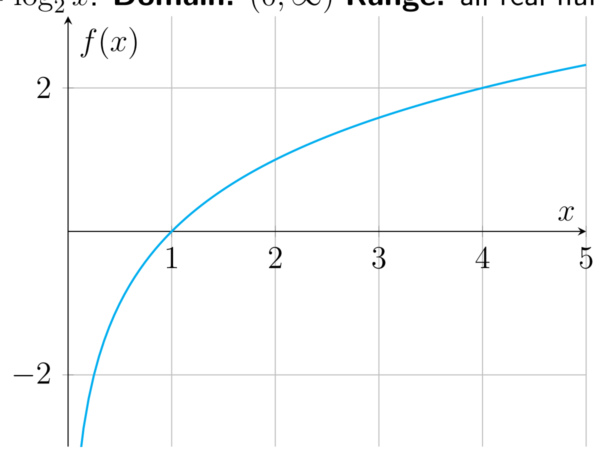
6. Logarithmic Functions

A **logarithmic function** is the inverse of an exponential function:

$$f(x) = \log_a x, \quad a > 0, a \neq 1$$

Example 9:

Consider $f(x) = \log_2 x$. **Domain:** $(0, \infty)$ **Range:** all real numbers $\mathbb{R}$

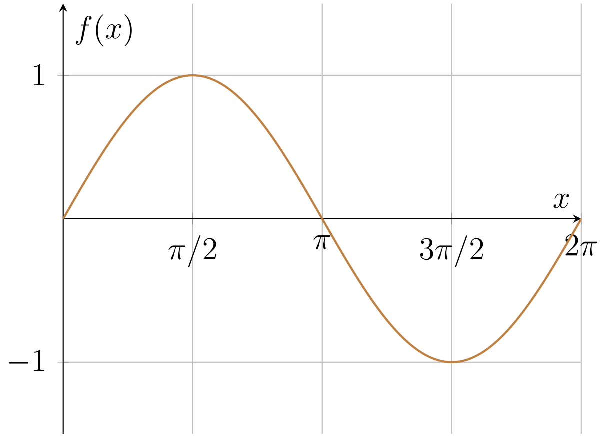


7. Trigonometric Functions

Trigonometric functions like sine, cosine, and tangent are fundamental periodic functions.

Example 10:

Consider $f(x) = \sin x$. **Domain:** all real numbers $\mathbb{R}$ **Range:** $[-1, 1]$



8. Absolute Value Function

Definition: Absolute Value Function

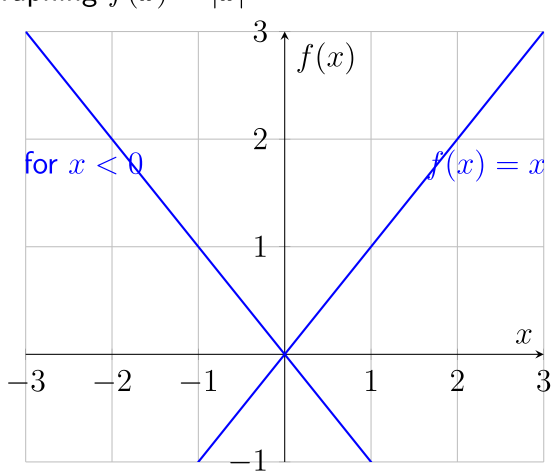
The **absolute value function** $f(x) = |x|$ is defined as

$$f(x) = \begin{cases} -x, & x < 0, \\ x, & x \geq 0. \end{cases}$$

- The function outputs the distance of x from 0 on the real number line.
- For negative inputs, it returns $-x$ (makes it positive).
- For zero and positive inputs, it returns x (same value).

Example 11:

Graphing $f(x) = |x|$



Properties of $|x|$

Properties

- $|x| \geq 0$ for all $x \in \mathbb{R}$.
- $|x| = 0$ if and only if $x = 0$.
- $|ab| = |a||b|$, $|a/b| = |a|/|b|$ for $b \neq 0$.
- $|a + b| \leq |a| + |b|$ (Triangle Inequality).

Example 12:

Evaluating $|x|$

$$f(-5) = |-5| = 5, \quad f(3) = |3| = 3, \quad f(0) = |0| = 0$$

1.3 Types of Functions

Functions can be classified based on how elements in the domain and range are related. Here are some important types:

Concept: Types of Functions

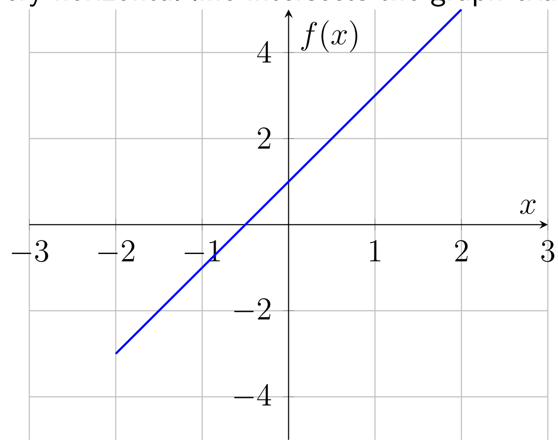
- **One-to-One (Injective)**: Each element of the domain maps to a *unique* element of the range.
- **Onto (Surjective)**: Every element of the range has at least one element of the domain mapping to it.
- **Bijective**: Function is both one-to-one and onto. Such functions have inverses.
- **Many-to-One**: Two or more elements of the domain map to the same element in the range.
- **Constant Function**: All inputs map to the same output.
- **Identity Function**: Each element maps to itself.

1. One-to-One Function (Injective)

Example 13:

One-to-One Function Consider $f(x) = 2x + 1$.
Domain: $\mathbb{R}$, **Range:** $\mathbb{R}$

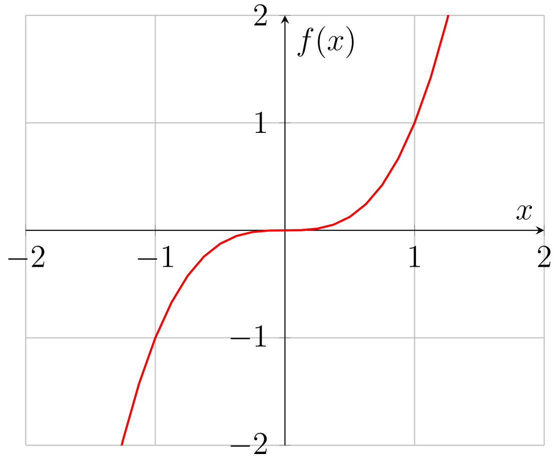
Check horizontal line test: every horizontal line intersects the graph exactly once, so f is one-to-one.



2. Onto Function (Surjective)

Example 14:

Onto Function Consider $f(x) = x^3$.
Domain: $\mathbb{R}$, **Range:** $\mathbb{R}$ Every $y \in \mathbb{R}$ has a solution $x = \sqrt[3]{y}$, so f is onto.

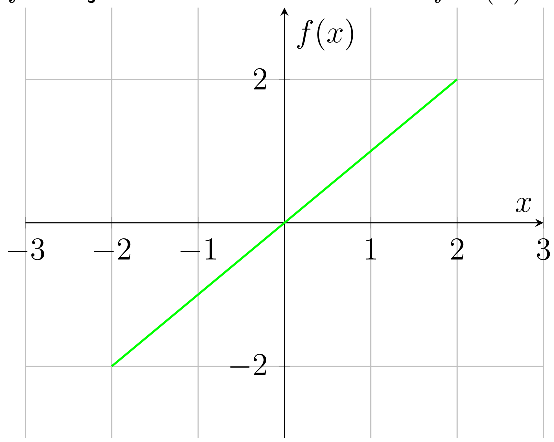


3. Bijective Function

Example 15:

Bijective Function Consider $f(x) = x + 3$.
- One-to-one: Yes, each input gives unique output. - Onto: Yes, every $y \in \mathbb{R}$ is covered.

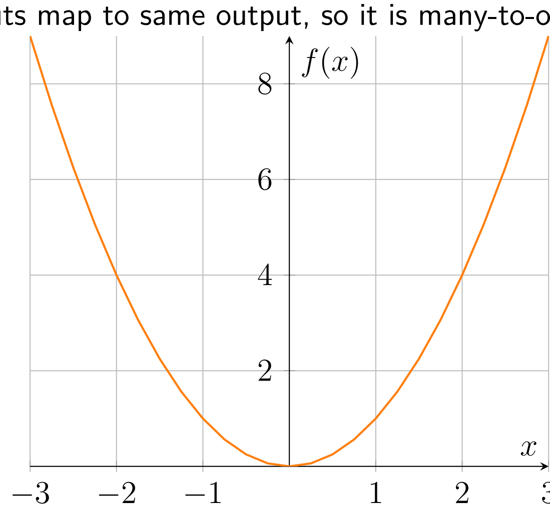
Hence, f is bijective and has an inverse $f^{-1}(x) = x - 3$.



4. Many-to-One Function

Example 16:

Many-to-One Function Consider $f(x) = x^2$.
- $f(2) = f(-2) = 4$ - Multiple inputs map to same output, so it is many-to-one. - Graph fails horizontal line test.

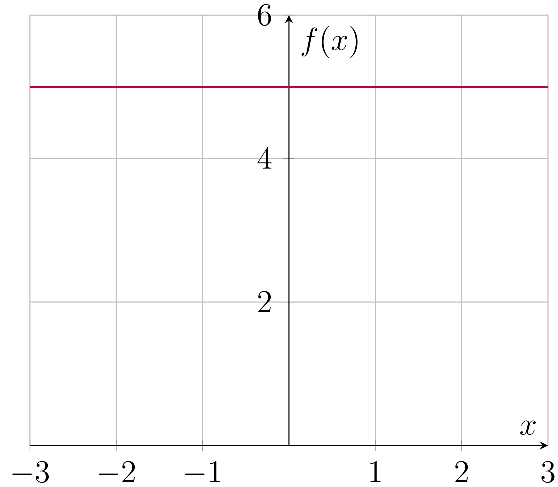


5. Constant Function

Example 17:

Constant Function Consider $f(x) = 5$.

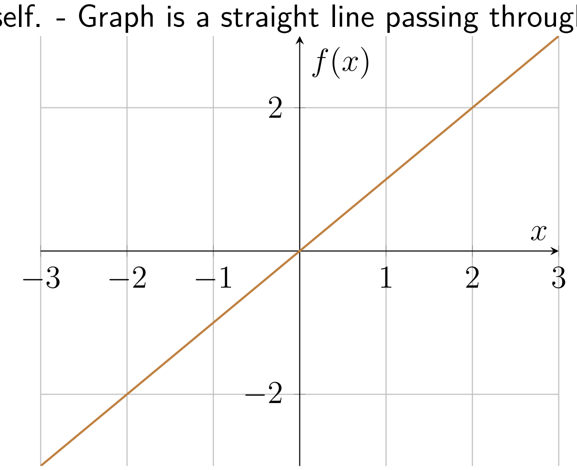
- All inputs map to the same output. - Graph is a horizontal line.



6. Identity Function

Example 18:

Identity Function Consider $f(x) = x$.
- Each input maps to itself. - Graph is a straight line passing through the origin with slope 1.



1.4 Inverse Function

An inverse function reverses the operation done by a particular function. Whatever a function does, the inverse function undoes it.

Definition: Inverse Function

Given a function f with domain D and range R, its **inverse function** (if it exists) is the function f^{-1} with domain R and range D such that

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y.$$

Equivalently,

$$f^{-1}(f(x)) = x \quad \text{for all } x \in D, \quad f(f^{-1}(y)) = y \quad \text{for all } y \in R.$$

Example of an Inverse Function

Example 19:

Finding an Inverse Let $f(x) = x^3 + 4$. Solve $y = x^3 + 4$ for x:

$$x = \sqrt[3]{y-4}$$

Define $f^{-1}(y) = \sqrt[3]{y-4}$.
Verification:

$$f^{-1}(f(x)) = \sqrt[3]{(x^3+4)-4} = x$$

One-to-One Functions

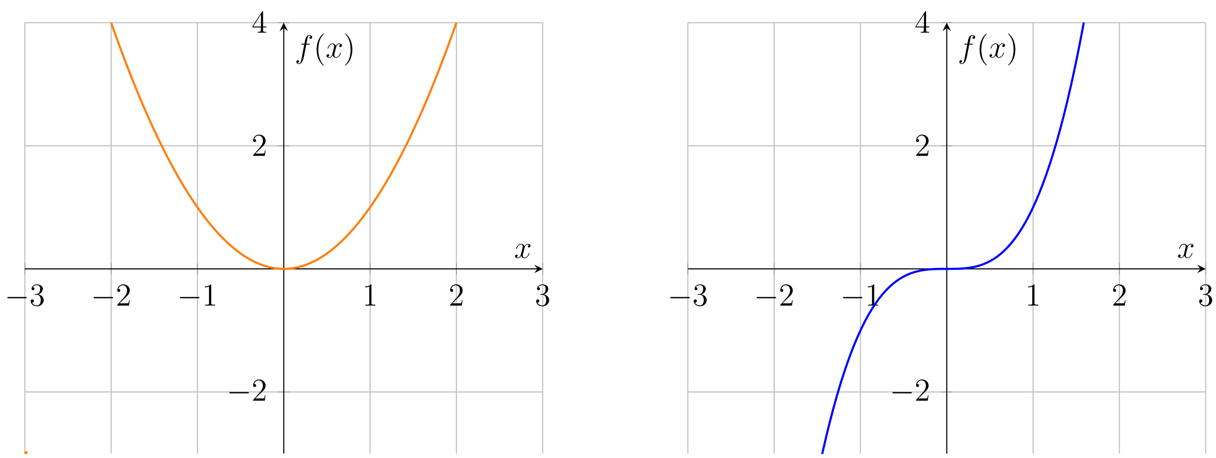
Definition: One-to-One Function

A function f is **one-to-one** if

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2$$

Horizontal Line Test: A function is one-to-one if every horizontal line intersects its graph at most once.

Example 20:



Finding an Inverse Function

Strategy: Solve $y = f(x)$ for x, then interchange x and y:

$$y = f^{-1}(x)$$

Example 21:

Finding an Inverse Find the inverse of $f(x) = 3x - 4$.
Solution:

$$y = 3x - 4 \implies x = \frac{y+4}{3} \implies f^{-1}(x) = \frac{x+4}{3}$$

Domain and Range: Both are $\mathbb{R}$. Verification:

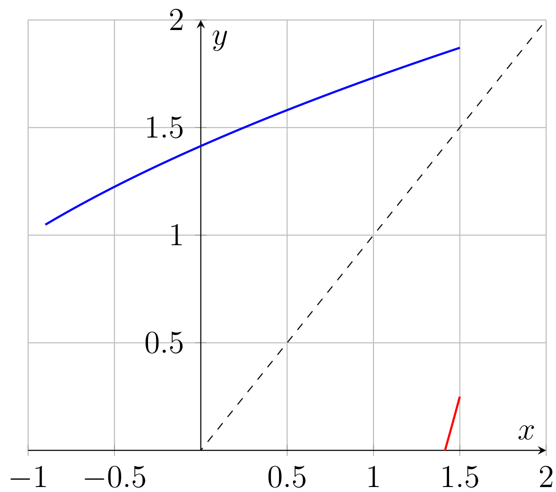
$$f^{-1}(f(x)) = \frac{3x - 4 + 4}{3} = x$$

Graphing Inverse Functions

Concept: Graph of Inverse Function

The graph of f^{-1} is the reflection of the graph of f about the line $y = x$.
If (a, b) is on f, then (b, a) is on f^{-1}.

Example 22:



Sketching Graph of an Inverse

Restricting Domains

Some functions are not one-to-one over their entire domain. By restricting the domain, we can define an inverse.

Example 23:

Restricting Domain of $f(x) = x^2$

$$g(x) = x^2, \quad x \geq 0 \implies g^{-1}(x) = \sqrt{x}, \quad \text{Domain } [0, \infty)$$

$$h(x) = x^2, \quad x \leq 0 \implies h^{-1}(x) = -\sqrt{x}, \quad \text{Domain } (-\infty, 0]$$

1.5 Composite Functions

Definition: Composite Function

Given two functions f and g, the **composite function** $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x))$$

for all x in the domain of g such that $g(x)$ lies in the domain of f.

Example of Composite Function

Example 24:

Composite Function Example Let $f(x) = x^2$ and $g(x) = x + 1$. Then,

$$(f \circ g)(x) = f(g(x)) = f(x + 1) = (x + 1)^2$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 1$$

Notice that in general, $f \circ g \neq g \circ f$.

Rules of Composite Functions

Rules of Composite Functions

1. **Associative Law:** $(f \circ g) \circ h = f \circ (g \circ h)$
2. **Domain Rule:** The domain of $f \circ g$ consists of all x in the domain of g such that $g(x)$ is in the domain of f.
3. **Non-commutative:** In general, $f \circ g \neq g \circ f$.

Example: Associative Law

Example 25:

Associativity Let $f(x) = x^2$, $g(x) = x + 1$, and $h(x) = 2x$.
Compute $(f \circ g) \circ h$:

$$(f \circ g) \circ h(x) = f(g(h(x))) = f((2x) + 1) = (2x + 1)^2$$

Compute $f \circ (g \circ h)$:

$$f \circ (g \circ h)(x) = f(g(h(x))) = f((2x) + 1) = (2x + 1)^2$$

Hence, $(f \circ g) \circ h = f \circ (g \circ h)$

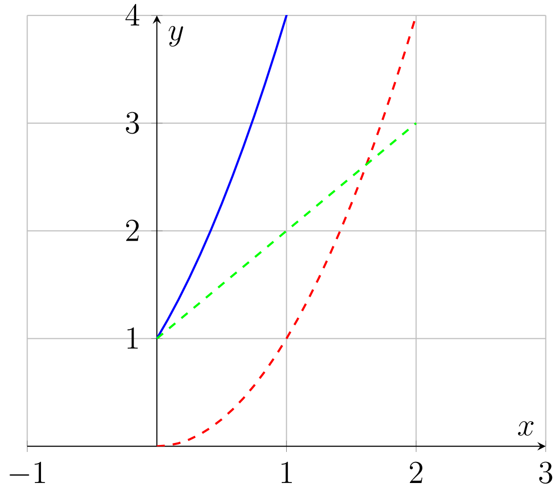
Domain of Composite Functions

Example 26:

Domain Example Let $f(x) = \sqrt{x}$, $g(x) = x - 1$.
Then $(f \circ g)(x) = f(g(x)) = \sqrt{x - 1}$.
Domain: $x - 1 \geq 0 \implies x \geq 1$, so the domain is $[1, \infty)$

Graphical Representation

Example 27:



Graph of Composite Function

- Blue: $(f \circ g)(x) = (x + 1)^2$
- Green dashed: $g(x) = x + 1$
- Red dashed: $f(x) = x^2$

1.6 Problems

Problem 1 ***Find the domain and range of the following functions:***

a) $Y(t) = 2t^2 - 3t + 4$

b) $f(z) = 1 + \sqrt{z^2 + 4}$

c) $M(x) = 7 - |x + 3|$

Problem 2 ***Find the domain of the following functions:***

a) $R(z) = 4z^3 + 6z^2 + 5z$

b) $g(x) = \sqrt{16 - x^2}$

c) $P(t) = \frac{4t+2}{\sqrt{t^3 - t^2 - 6t}}$

d) $A(x) = 3x - 7 - \sqrt{x^2 - 25}$

Problem 3 *Find the domain and range of the following functions:*

a) $f(x) = e^x - 3$

b) $g(x) = \ln(x - 1)$

c) $h(x) = \cos x$

Problem 4 ***Compute composite functions*** $(f \circ g)(x)$ ***and*** $(g \circ f)(x)$***:***

a) $f(x) = 3x - 2,\ g(x) = \sqrt{5 + 4x}$

b) $f(x) = x^2 - x + 2,\ g(x) = 7 - 2x^2$

Problem 5 ***Find the inverse of the following functions and verify the composition:***

a) $f(x) = 5x + 10$

b) $R(x) = x^3 + 5$

c) $W(x) = 4\sqrt{10 - 9x}$

1.7 Try it Yourself

Exercise 1 ***Evaluate the following functions:***

$$f(x) = 2 - 3x - x^2, \quad g(t) = \frac{t^2}{t + 5}, \quad h(z) = \sqrt{2 - z^2}, \quad R(x) = \sqrt{4 + x} - 3x + 2$$

a) $f(-2)$

b) $f(5 - 3x)$

c) $g(-2)$

d) $g(x^2)$

e) $g(t + h)$

f) $g(t^2 - 2t + 1)$

g) $h(-1/3)$

h) $h(2z)$

i) $h(z^2 - z)$

j) $h(z + k)$

k) $R(5)$

l) $R(x + 2)$

m) $R(x^3 - 2)$

n) $R\left(\frac{2}{x} - 1\right)$

Exercise 2 ***Determine all the roots of the following functions:***

a) $R(y) = 10y^2 + 9y - 4$

b) $W(x) = x^4 + 5x^2 - 24$

c) $h(z) = z^3 - 4z - 6$

d) $g(w) = \frac{3w}{w+2} + w - \frac{5}{2w-4}$

Exercise 3 ***Find the domain and range of the following functions:***

a) $g(z) = -z^2 - 3z + 6$

b) $h(y) = -2\sqrt{12 + 4y}$

Exercise 4 ***Find the domain of the following functions:***

a) $f(w) = \frac{w^3-2w+1}{10w-5}$

b) $g(t) = \frac{5t-t^3}{6-t-3t^2}$

c) $h(x) = \sqrt{x^4 - x^3 - 16x^2}$

d) $f(z) = \sqrt{z-2} + \sqrt{z+5}$

e) $h(y) = \frac{\sqrt{3y+8}-2}{\sqrt{3-y}}$

f) $Q(y) = \sqrt{y^2+2} - 2\sqrt{1-y}$

Exercise 5 ***Compute composite functions*** $(f \circ g)(x)$ ***and*** $(g \circ f)(x)$***:***

a) $f(x) = 4x + 3,\ g(x) = x^2 - 10x$

b) $f(x) = x^2 + 4,\ g(x) = \sqrt{6 + x^2}$

Exercise 6 ***Find the inverse of the following functions and verify the composition:***

a) $h(x) = 5 - 17x$

b) $g(x) = 3(x-2)^4 + 15$

c) $f(x) = 6\sqrt{4x+7}$

d) $h(x) = 2 + \frac{8}{x^3-x}$

e) $f(x) = 5 - 8x^{6x} + 3$

1.8 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
1	Chapter 1.1–1.3: Introduction to Functions — Domain, Range, Codomain, Classes. *GATE DA Calculus & Optimization Lecture 1: Functions (Ch 1.1–1.3)*	https://youtu.be/MVTL8oUWRFY	
2	Chapter 1.4–1.5: Types, Inverse, Composite. *GATE DA Calculus & Optimization Lecture 2: Functions (Ch 1.4–1.5)*	https://youtu.be/u7jztXBIu9s	
3	Chapter 1.4–1.5: Solutions to Problems 1–5. *GATE DA Calculus & Optimization Lecture 3: Solutions*	https://youtu.be/OwEf3BBTK9A	

Chapter 2

Limits

2.1 Introduction to Calculus

Calculus emerged from the need to solve practical problems in physics and engineering. Two fundamental problems led to its development:

1. **The Tangent Problem**: Determining the slope of a line tangent to a curve at a given point.
2. **The Area Problem**: Calculating the area under a curve.

The Tangent Problem and Differential Calculus

The rate of change is a critical concept in calculus. To understand this, consider the graphs of the following linear functions:

- $f(x) = -2x - 3$
- $g(x) = \frac{x}{2} + 1$
- $h(x) = 2$

Concept

The slope of a linear function indicates the rate of change. For example:

- $f(x) = -2x - 3$: slope = -2
- $g(x) = \frac{x}{2} + 1$: slope = 1/2
- $h(x) = 2$: slope = 0

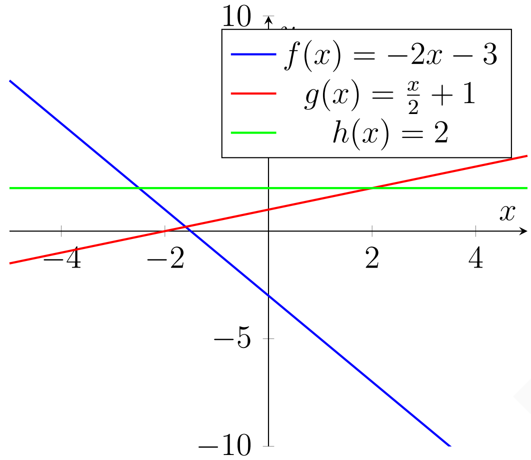


Figure 2.1: Graphs of linear functions with constant rates of change.

Now consider the quadratic function $k(x) = x^2$. Its rate of change is not constant. To approximate it, we use secant lines.

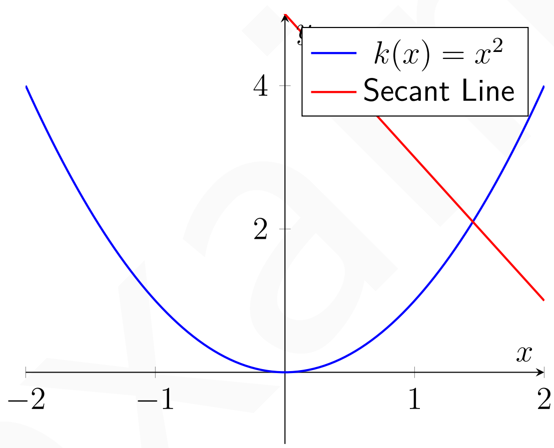


Figure 2.2: Graph of $k(x) = x^2$ with a secant line.

Definition

A **secant line** to a function $f(x)$ through points $(a, f(a))$ and $(x, f(x))$ passes through these points. Its slope is

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

As x approaches a, the secant line approaches the tangent line. Its slope approaches the derivative of $f(x)$ at a.

Example 28:

Find the slope of the secant line to $k(x) = x^2$ through points $(1, 1)$ and $(2, 4)$:

$$m_{\text{sec}} = \frac{4 - 1}{2 - 1} = 3.$$

Concept

The slope of the tangent line at a point gives the instantaneous rate of change, which is the derivative of the function at that point.

The Area Problem and Integral Calculus

To find the area under a curve, approximate by summing rectangles.

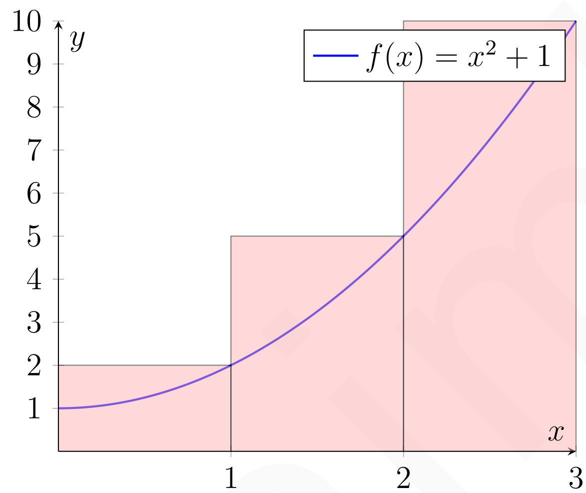


Figure 2.3: Approximation of the area under $f(x) = x^2 + 1$ using rectangles.

Definition

The **area under a curve** is approximated by summing the areas of rectangles over subintervals. As the width of the rectangles decreases, the sum approaches the exact area. This process leads to the definite integral.

Example 29:

Estimate the area under $f(x) = x^2 + 1$ over $[0, 3]$ using three rectangles:

$$\text{Area} \approx \sum_{i=1}^{3} \text{Area of Rectangle}_i$$

Conclusion

The Tangent and Area Problems lead to the foundations of calculus:

- Tangent Problem $\rightarrow$ Derivative (rate of change)
- Area Problem $\rightarrow$ Integral (area under curve)

These concepts are essential for further studies in calculus and multivariable calculus.

2.2 Limits

Definition of a Limit

We say the **limit** of $f(x)$ is L as x approaches a and write

$$\lim_{x \to a} f(x) = L$$

provided we can make $f(x)$ as close to L as we want for all x sufficiently close to a, from both sides, without actually letting $x = a$.

Intuitive Idea

The limit describes the behavior of $f(x)$ around $x = a$, not necessarily the value at $x = a$. As x approaches a, $f(x)$ must move closer and closer to L. This must hold for values of x on both sides of a.

Estimating Limits

To estimate limits, we examine the behavior of a function as x approaches a point a from both sides. The idea is to find a value X such that whenever $|x - a| < X$, the function's value $f(x)$ stays within a desired tolerance of the limit L.

Example 30:

Estimate

$$\lim_{x \to 3} \frac{x^2 + 5x - 18}{x^2 - 3x}.$$

Observation

Although the function is undefined at $x = 3$, we can study its behavior for values near 3.

Example 31:

Define

$$g(x) = \begin{cases} \dfrac{x^2 + 5x - 18}{x^2 - 3x}, & x \neq 3 \\ 7, & x = 3 \end{cases}$$

Estimate $\lim_{x \to 3} g(x)$.

Answer: As in Example 1, the limit is 4, even though $g(3) = 7$. This shows that limits depend on nearby values of x, not the function's value at the point itself.

Example 32:

Estimate

$$\lim_{\theta \to 0} \frac{1 - \cos(2\theta)}{\theta}.$$

Observation: The function is undefined at $\theta = 0$, but by evaluating values close to 0, we can approximate the limit.

Example 33:

Estimate

$$\lim_{t \to 0} \sin(\pi t/2).$$

Observation: Tables of values may help, but they can be misleading. Graphical analysis or algebraic methods provide more reliable estimates.

Example 34:

Estimate

$$\lim_{t \to 0} X(t), \quad X(t) = \begin{cases} -1, & t < 0 \\ 2, & t \geq 0 \end{cases}$$

Observation: Not all limits exist. Here, the left-hand limit is -1 and the right-hand limit is 2, so the limit does not exist.

Summary

Key Points

- Limits describe the behavior of a function around a point, not necessarily the function's value at that point.
- Limits can exist even if the function does not exist at that point.
- A function can exist at a point, but its limit at that point might be different.
- Tables and graphs can help understand limits but are not always reliable for exact values.
- Some limits do not exist.

2.3 One-Sided Limits

In the previous section, we saw two limits that did not exist for different reasons:

- $\lim_{t \to 0} \cos(\pi t)$ does not exist because the function oscillates and does not settle to a single value near $t = 0$.
- $\lim_{t \to 0} X(t), \quad X(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$ does not exist because the left-hand and right-hand limits are different.

To distinguish these situations, we introduce **one-sided limits**.

Definitions: One-Sided Limits

Right-Hand Limit

We say

$$\lim_{x \to a^+} f(x) = L$$

if we can make $f(x)$ as close to L as desired for all $x > a$ sufficiently close to a, without considering $x = a$.

Left-Hand Limit

We say

$$\lim_{x \to a^-} f(x) = L$$

if we can make $f(x)$ as close to L as desired for all $x < a$ sufficiently close to a, without considering $x = a$.

Observation

The only difference between one-sided limits and ordinary limits is the direction from which x approaches a. For a one-sided limit to exist, the function must approach a unique value from that side.

Examples

Example 35:

Estimate:

$$\lim_{t \to 0^+} H(t) \quad \text{and} \quad \lim_{t \to 0^-} H(t), \quad H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Solution:

$$\lim_{t \to 0^+} H(t) = 1, \quad \lim_{t \to 0^-} H(t) = 0$$

Example 36:

Estimate:

$$\lim_{t \to 0^+} \sin\left(\frac{1}{t}\right) \quad \text{and} \quad \lim_{t \to 0^-} \sin\left(\frac{1}{t}\right)$$

Observation: The function oscillates infinitely near $t = 0$, so the one-sided limits do not exist.

Example 37:

Let

$$g(x) = \begin{cases} \dfrac{x^2 + 5x - 18}{x^2 - 3x}, & x \neq 3 \\ 7, & x = 3 \end{cases}$$

Estimate:

$$\lim_{x \to 3^+} g(x) \quad \text{and} \quad \lim_{x \to 3^-} g(x)$$

Solution: Both one-sided limits exist and equal 4.

Fact Relating One-Sided and Ordinary Limits

Fact

For a function $f(x)$:

- If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$, then $\lim_{x \to a} f(x) = L$.
- Conversely, if $\lim_{x \to a} f(x) = L$, both one-sided limits exist and equal L.
- If $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

2.4 Properties of Limits

Properties of Limits

Let $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, and let c be any constant. Then the following properties hold:

1. Constant Multiple:

$$\lim_{x \to a} [c f(x)] = c \lim_{x \to a} f(x)$$

Example: $\lim_{x \to 2} 4x^2 = 4 \cdot 4 = 16$

2. Sum/Difference:

$$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

Example: $\lim_{x \to 1} (x^2 + 3x) = 1 + 3 = 4$

3. Product:

$$\lim_{x \to a} [f(x) g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

Example: $\lim_{x \to 2} (x^2 \cdot (x+1)) = 4 \cdot 3 = 12$

4. Quotient:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \quad \lim_{x \to a} g(x) \neq 0$$

Example: $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4$

5. Power:

$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$$

Example: $\lim_{x\to 3}(x-1)^3 = 2^3 = 8$

6. Root:

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$

Example: $\lim_{x\to 9} \sqrt{x} = 3$

7. Constant Function: $\lim_{x\to a} c = c$

8. Identity Function: $\lim_{x\to a} x = a$

9. Power of Identity: $\lim_{x\to a} x^n = a^n$

2.5 Infinite Limits

In this section we study limits whose value is ∞ or $-\infty$. These limits appear frequently in calculus and other exams, so understanding them is essential.

Definition of Infinite Limits

We say

$$\lim_{x\to a} f(x) = \infty$$

if we can make $f(x)$ arbitrarily large for all x sufficiently close to a (from both sides) without actually letting $x = a$.
Similarly,

$$\lim_{x\to a} f(x) = -\infty$$

if we can make $f(x)$ arbitrarily large and negative for all x sufficiently close to a without letting $x = a$.
These definitions can be extended to one-sided limits as well.

Vertical Asymptote

A function $f(x)$ has a vertical asymptote at $x = a$ if any of the following limits is true:

$$\lim_{x\to a^-} f(x) = \pm\infty, \quad \lim_{x\to a^+} f(x) = \pm\infty, \quad \lim_{x\to a} f(x) = \pm\infty$$

Note: Only one of the above needs to hold.

Facts About Infinite Limits

Let $f(x)$ and $g(x)$ be functions such that

$$\lim_{x\to c} f(x) = \infty, \quad \lim_{x\to c} g(x) = L$$

Then:

$$\lim_{x\to c}[f(x) \pm g(x)] = \infty$$

$$\lim_{x\to c}[f(x)g(x)] = \begin{cases} \infty & \text{if } L > 0 \\ -\infty & \text{if } L < 0 \end{cases}$$

$$\lim_{x\to c}\frac{g(x)}{f(x)} = 0$$

Example 38:

Evaluate the following limits:

$$\lim_{x\to 0^+}\frac{1}{x}, \quad \lim_{x\to 0^-}\frac{1}{x}, \quad \lim_{x\to 0}\frac{1}{x}$$

Solution: - As $x \to 0^+$, $x > 0$ so $\frac{1}{x} \to +\infty$. - As $x \to 0^-$, $x < 0$ so $\frac{1}{x} \to -\infty$. - Since the left and right limits are not equal, $\lim_{x\to 0}\frac{1}{x}$ does not exist.

Example 39:

Evaluate the following limits:

$$\lim_{x\to 0^+}\frac{6}{x^2}, \quad \lim_{x\to 0^-}\frac{6}{x^2}, \quad \lim_{x\to 0}\frac{6}{x^2}$$

Solution: - As $x \to 0$, $x^2 > 0$ so $\frac{6}{x^2} \to +\infty$ from both sides. - Therefore, $\lim_{x\to 0^+}\frac{6}{x^2} = \lim_{x\to 0^-}\frac{6}{x^2} = \lim_{x\to 0}\frac{6}{x^2} = +\infty$

Example 40:

Evaluate the following limits:

$$\lim_{x\to -2^+}\frac{-4}{x+2}, \quad \lim_{x\to -2^-}\frac{-4}{x+2}, \quad \lim_{x\to -2}\frac{-4}{x+2}$$

Solution: - As $x \to -2^+$, $x + 2 > 0$ so $\frac{-4}{x+2} \to -\infty$. - As $x \to -2^-$, $x + 2 < 0$ so $\frac{-4}{x+2} \to +\infty$. - Left and right limits are not equal, so the limit at $x = -2$ does not exist.

Example 41:

Evaluate the following limits:

$$\lim_{x\to 4^+}\frac{3}{(4-x)^3}, \quad \lim_{x\to 4^-}\frac{3}{(4-x)^3}, \quad \lim_{x\to 4}\frac{3}{(4-x)^3}$$

Solution: - As $x \to 4^+$, $4 - x < 0$, so $(4-x)^3 < 0$ and $\frac{3}{(4-x)^3} \to -\infty$. - As $x \to 4^-$, $4 - x > 0$, so $\frac{3}{(4-x)^3} \to +\infty$. - Since left and right limits differ, $\lim_{x\to 4}\frac{3}{(4-x)^3}$ does not exist.

Example 42:

Evaluate the following limits:

$$\lim_{x\to 3^+} \frac{2x}{x-3}, \quad \lim_{x\to 3^-} \frac{2x}{x-3}, \quad \lim_{x\to 3} \frac{2x}{x-3}$$

Solution: - As $x \to 3^+$, $x-3>0$ so $\frac{2x}{x-3} \to +\infty$. - As $x \to 3^-$, $x-3<0$ so $\frac{2x}{x-3} \to -\infty$. - Left and right limits differ, so $\lim_{x\to 3} \frac{2x}{x-3}$ does not exist.

Example 43:

Evaluate the one-sided limit:

$$\lim_{x\to 0^+} \ln x$$

Solution: - As $x \to 0^+$, $\ln x \to -\infty$. - So, $\lim_{x\to 0^+} \ln x = -\infty$

Example 44:

Evaluate the following limits:

$$\lim_{x\to \pi/2^+} \tan x, \quad \lim_{x\to \pi/2^-} \tan x$$

Solution: - As $x \to \pi/2^-$, $\tan x \to +\infty$ - As $x \to \pi/2^+$, $\tan x \to -\infty$

Example 45:

Show that a vertical asymptote exists at $x = 0$ for $f(x) = 1/x$.
Solution: - One-sided limits: $\lim_{x\to 0^+} 1/x = +\infty$, $\lim_{x\to 0^-} 1/x = -\infty$ - Since at least one one-sided limit goes to $\pm\infty$, $x = 0$ is a vertical asymptote.

Example 46:

Evaluate the following limit:

$$\lim_{x\to 0} \frac{5x+1}{x^2}$$

Solution: - Split: $\frac{5x+1}{x^2} = \frac{5x}{x^2} + \frac{1}{x^2} = \frac{5}{x} + \frac{1}{x^2}$ - As $x \to 0^+$, $\frac{5}{x} + \frac{1}{x^2} \to +\infty$ - As $x \to 0^-$, $\frac{5}{x} + \frac{1}{x^2} \to -\infty + \infty = +\infty$ (dominant term $1/x^2$) - So, $\lim_{x\to 0} \frac{5x+1}{x^2} = +\infty$

Example 47:

Evaluate the following limit:

$$\lim_{x\to -1^+} \frac{2}{x+1}, \quad \lim_{x\to -1^-} \frac{2}{x+1}$$

Solution: - As $x \to -1^+$, $x+1>0 \implies \frac{2}{x+1} \to +\infty$ - As $x \to -1^-$, $x+1<0 \implies \frac{2}{x+1} \to -\infty$

2.6 Limits at Infinity

By limits at infinity, we study the behavior of functions as x becomes very large:

$$\lim_{x\to\infty} f(x) \quad \text{or} \quad \lim_{x\to-\infty} f(x)$$

These limits may also be ∞ or $-\infty$.

Fact 1: Constant over Power

If $r > 0$ is rational and c is a constant:

$$\lim_{x\to\infty} \frac{c}{x^r} = 0, \quad \lim_{x\to-\infty} \frac{c}{x^r} = 0 \quad (\text{if } x^r \text{ defined for } x < 0)$$

Explanation: As $x \to \infty$, x^r grows, making the fraction approach 0. The sign of c only affects the approach from above or below.

Fact 2: Polynomials at Infinity

For $p(x) = a_n x^n + \cdots + a_0$, degree n, $a_n \neq 0$:

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n, \quad \lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_n x^n$$

Explanation: Highest-degree term dominates; lower-degree terms are negligible.

Horizontal Asymptote

A function $f(x)$ has a horizontal asymptote at $y = L$ if:

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

Explanation: The graph approaches $y = L$ as x moves far from the origin.

Example 48:

$$\lim_{x\to\infty} (2x^4 - x^2 - 8x), \quad \lim_{t\to-\infty} \left(\frac{1}{3}t^5 + 2t^3 - t^2 + 8\right)$$

Solution:

- $\lim_{x\to\infty}(2x^4 - x^2 - 8x) = \infty$ (dominant $2x^4$).
- $\lim_{t\to-\infty}(\frac{1}{3}t^5 + \dots) = -\infty$ (dominant $\frac{1}{3}t^5$ negative).

Example 49:

$$\lim_{x\to\pm\infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7}$$

Solution: Factor x^4:

$$\frac{2 - 1/x^2 + 8/x^3}{-5 + 7/x^4} \to -\frac{2}{5} \quad \text{as } x \to \pm\infty$$

Example 50:

$$\lim_{x\to\infty} \frac{\sqrt{3x^2+6}}{5-2x}, \quad \lim_{x\to-\infty} \frac{\sqrt{3x^2+6}}{5-2x}$$

Solution:

- $x \to \infty : \frac{\sqrt{x^2(3+6/x^2)}}{x(-2+5/x)} \to -\frac{\sqrt{3}}{2}$
- $x \to -\infty : \frac{-x\sqrt{3+6/x^2}}{x(-2+5/x)} \to \frac{\sqrt{3}}{2}$

Exponential Function Behavior

$$\lim_{x\to\infty} e^x = \infty, \quad \lim_{x\to-\infty} e^x = 0, \quad \lim_{x\to\infty} e^{-x} = 0, \quad \lim_{x\to-\infty} e^{-x} = \infty$$

Example 51:

$$\lim_{x\to\infty} e^{2-4x-8x^2} = 0, \quad \lim_{t\to-\infty} e^{t^4-5t^2+1} = \infty, \quad \lim_{z\to0^+} e^{1/z} = \infty$$

Example 52:

$$\lim_{x\to\infty} \left(e^{10x} - 4e^{6x} + \dots\right) = \infty, \quad \lim_{x\to-\infty} \left(e^{10x} - 4e^{6x} + \dots\right) = -\infty$$

Logarithmic Function Limits

$$\lim_{x\to0^+} \ln x = -\infty, \quad \lim_{x\to\infty} \ln x = \infty$$

Example 53:

$$\lim_{x\to\infty} \ln(7x^3 - x^2 + 1) = \infty, \quad \lim_{t\to-\infty} \ln(t^2 - 5t) = \infty$$

Inverse Tangent Function Limits

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}, \quad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

Example 54:

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}, \quad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}, \quad \lim_{x\to\infty} \tan^{-1}(x^3-5x+6) = \frac{\pi}{2}, \quad \lim_{x\to0^-} \tan^{-1}\left(\frac{1}{x}\right) = -\frac{\pi}{2}$$

2.7 Limit Formulas

Basic Limit Properties

- $\lim_{x\to a} c = c$
- $\lim_{x\to a} x = a$
- $\lim_{x\to a} \big(f(x) \pm g(x)\big) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a} \big(f(x)g(x)\big) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
- $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \ g(a) \neq 0$

Limits Involving Infinity (Rational Functions)

- $\lim_{x\to\infty} \dfrac{a_n x^n + \cdots}{b_m x^m + \cdots} = \begin{cases} 0, & n < m \\ \dfrac{a_n}{b_m}, & n = m \\ \infty \text{ or } -\infty, & n > m \end{cases}$
- $\lim_{x\to a^-} \dfrac{1}{x-a} = -\infty$
- $\lim_{x\to a^+} \dfrac{1}{x-a} = +\infty$

Standard Trigonometric Limits

- $\lim_{x\to0} \dfrac{\sin x}{x} = 1$
- $\lim_{x\to0} \dfrac{\tan x}{x} = 1$
- $\lim_{x\to0} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$

Exponential and Logarithmic Limits

- $\lim_{x\to0} \dfrac{e^x - 1}{x} = 1$
- $\lim_{x\to0} \dfrac{\ln(1+x)}{x} = 1$

- $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$
- $\lim_{x\to 0^+} x^n \ln x = 0 \quad (n > 0)$
- $\lim_{x\to\infty}\frac{\ln x}{x} = 0$
- $\lim_{x\to\infty}\frac{x^n}{e^x} = 0$

Inverse Trigonometric Limits

- $\lim_{x\to\infty}\arctan x = \frac{\pi}{2}$
- $\lim_{x\to-\infty}\arctan x = -\frac{\pi}{2}$
- $\lim_{x\to 0}\frac{\arcsin x}{x} = 1$
- $\lim_{x\to 0}\frac{\arctan x}{x} = 1$

L'Hospital's Rule

- If $\lim_{x\to a}\frac{f(x)}{g(x)}$ is $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}, \quad \text{if the limit exists.}$$

Squeeze Theorem

- If $f(x) \le g(x) \le h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$

Other Useful Limits

- $\lim_{x\to 0^+} x^x = 1$
- $\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{mx} = e^{km}, \quad k, m \in \mathbb{R}$
- $\lim_{x\to\infty} e^{-x} = 0$
- $\lim_{x\to-\infty} e^{x} = 0$
- $\lim_{x\to\infty} e^{x} = \infty$
- $\lim_{x\to-\infty} e^{-x} = \infty$

Example 55:

Evaluate $\lim_{x\to 2}(3x^2+5x-1)$.
Solution: Using the sum and constant multiple properties of limits:

$$\lim_{x\to 2}(3x^2+5x-1) = 3\lim_{x\to 2}x^2 + 5\lim_{x\to 2}x - 1 = 3(4)+5(2)-1 = 21$$

Example 56:

Evaluate $\lim_{x\to -1}\frac{x^3+1}{x+1}$.
Solution: Factor numerator: $x^3+1 = (x+1)(x^2-x+1)$

$$\lim_{x\to -1}\frac{(x+1)(x^2-x+1)}{x+1} = \lim_{x\to -1}(x^2-x+1) = 3$$

Example 57:

Evaluate $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$.
Solution: Rationalize the numerator:

$$\frac{\sqrt{x+4}-2}{x}\cdot\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \frac{1}{\sqrt{x+4}+2} \implies \lim_{x\to 0}\frac{1}{4} = \frac{1}{4}$$

Example 58:

Evaluate $\lim_{x\to 1}\frac{x^2-1}{x-1}$.
Solution: Factor numerator: $x^2-1 = (x-1)(x+1)$

$$\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) = 2$$

Example 59:

Evaluate $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ using $1-\cos x = 2\sin^2(x/2)$.
Solution:

$$\lim_{x\to 0}\frac{2\sin^2(x/2)}{x^2} = \lim_{x\to 0}2\left(\frac{\sin(x/2)}{x}\right)^2 = 2\left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

Example 60:

Evaluate $\lim_{x\to 4}\sqrt{x}$.
Solution: Using the root property:

$$\lim_{x\to 4}\sqrt{x} = \sqrt{\lim_{x\to 4}x} = 2$$

Example 61:

Evaluate $\lim_{x\to 0} \frac{\sin 3x}{x}$.
Solution: Factor 3:

$$\lim_{x\to 0} \frac{\sin 3x}{x} = 3 \lim_{x\to 0} \frac{\sin 3x}{3x} = 3(1) = 3$$

Example 62:

Evaluate $\lim_{x\to 0} \frac{\tan x}{x}$.
Solution: Standard limit property:

$$\lim_{x\to 0} \frac{\tan x}{x} = 1$$

Example 63:

Evaluate $\lim_{x\to 1} \frac{x^4-1}{x-1}$.
Solution: Factor numerator: $x^4 - 1 = (x-1)(x^3 + x^2 + x + 1)$

$$\lim_{x\to 1} \frac{(x-1)(x^3 + x^2 + x + 1)}{x-1} = \lim_{x\to 1}(x^3 + x^2 + x + 1) = 4$$

Example 64:

Evaluate $\lim_{x\to 2} \frac{x^2-4}{x^2+4}$.
Solution: Factor numerator: $x^2 - 4 = (x-2)(x+2)$

$$\lim_{x\to 2} \frac{(x-2)(x+2)}{x^2+4} = \frac{4}{8} = \frac{1}{2}$$

Example 65:

Evaluate $\lim_{x\to 0} \frac{e^x-1}{x}$.
Solution: Standard exponential limit:

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

Example 66:

Evaluate $\lim_{x\to 0} \frac{\ln(1+x)}{x}$.
Solution: Standard logarithmic limit:

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

Example 67:

Evaluate $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$.
Solution: Rationalize numerator:

$$\frac{\sqrt{x}-1}{x-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} = \frac{1}{\sqrt{x}+1} \implies \lim_{x\to 1} \frac{1}{2} = \frac{1}{2}$$

Example 68:

Evaluate $\lim_{x\to 0} \frac{\sin x + \tan x}{x}$.
Solution: Split limit:

$$\lim_{x\to 0} \frac{\sin x}{x} + \lim_{x\to 0} \frac{\tan x}{x} = 1 + 1 = 2$$

Example 69:

Evaluate $\lim_{x\to 0^+} \ln x$.
Solution: As $x \to 0^+$, $\ln x \to -\infty$

Example 70:

Evaluate $\lim_{x\to\infty} \frac{3x^2+2}{5x^2-7}$.
Solution: Divide numerator and denominator by x^2:

$$\lim_{x\to\infty} \frac{3+2/x^2}{5-7/x^2} = \frac{3}{5}$$

Example 71:

Evaluate $\lim_{x\to 0} \frac{1}{x}$.
Solution: Limit does not exist; as $x \to 0^+$, it $\to \infty$; as $x \to 0^-$, it $\to -\infty$

Example 72:

Evaluate $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x}$.
Solution: Rewrite:

$$\lim_{x\to 0} \frac{2x}{3x} \cdot \frac{\sin 2x/(2x)}{\sin 3x/(3x)} = \frac{2}{3} \cdot 1 = \frac{2}{3}$$

Example 73:

Evaluate $\lim_{x\to 1}(x^5-1)$.
Solution: Direct substitution:

$$\lim_{x\to 1}(x^5-1) = 1 - 1 = 0$$

Example 74:

Evaluate $\lim_{x\to 0} \frac{e^{2x}-1}{\sin x}$.
Solution: Split and use standard limits:

$$\lim_{x\to 0} \frac{e^{2x}-1}{x} \cdot \frac{x}{\sin x} = 2 \cdot 1 = 2$$

2.8 Problems

Problem 6 *Evaluate*

$$\lim_{x\to 0} \frac{\ln(1+2x) - 2\ln(1+x)}{x^2}.$$

Problem 7 *Compute*

$$\lim_{x\to\infty} \left(\frac{2x^2+3}{3x^2+2}\right)^x.$$

Problem 8 *Evaluate*

$$\lim_{x\to 0} \frac{\cos(x) - e^{-x^2/2}}{x^4}.$$

Problem 9 *Compute*

$$\lim_{x\to \pi/2^-} (\sec x - \tan x).$$

Problem 10 *Evaluate*

$$\lim_{x\to 0} \frac{e^{\sin x} - \cos x}{x}.$$

Problem 11 *Find*

$$\lim_{x\to 0} \frac{\sqrt{1+3x} - \sqrt{1-2x}}{x}.$$

Problem 12 *Compute*

$$\lim_{x\to 0} \frac{\ln(\cos x)}{x^2}.$$

Problem 13 *Evaluate*

$$\lim_{x\to 1} \frac{x^x - 1}{x - 1}.$$

Problem 14 *Find*

$$\lim_{x\to 0} \frac{e^{2x} + e^{3x} - 2}{x}.$$

Problem 15 *Compute*

$$\lim_{x\to\infty} \left(1 + \frac{1}{\sqrt{x}}\right)^{\sqrt{x}}.$$

Problem 16 *Find*

$$\lim_{x\to 0} \frac{1 - \cos(2x)}{x \sin x}.$$

Problem 17 *Evaluate*

$$\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x.$$

Problem 18 *Evaluate*

$$\lim_{x\to\pi/4}\frac{\tan x-1}{x-\pi/4}.$$

Problem 19 *Compute*

$$\lim_{x\to 0}\frac{\sqrt{1+x^2}-\cos x}{x^2}.$$

2.9 Try it Yourself

Exercise 7 *Evaluate the following limits, if they exist:*

(a) $\lim_{t\to -4}\frac{8+5t}{t^2+2}$

(b) $\lim_{x\to -6}\frac{x^2-36}{x^2+3x-18}$

(c) $\lim_{z\to 9}\frac{3z^2-20z+9}{9-z}$

(d) $\lim_{y\to 8}\frac{y^2-5y-24}{4y^2-19y-30}$

(e) $\lim_{h\to 0}\frac{(5+h)^2-25}{h}$

(f) $\lim_{z\to 9}\frac{\sqrt{z}-3}{z-9}$

(g) $\lim_{x\to -4}\frac{\sqrt{3x+26}-5}{x+4}$

(h) $\lim_{x\to 0}(x^3-\sqrt[3]{x}+7)$

(i) $\lim_{x\to 3}(7-4x+10x^2)$

Exercise 8 *Evaluate the following limits:*

(a) *For* $g(z)=2z+4(z+2)^2$, *find*

$$\lim_{z\to -2^-} g(z),\quad \lim_{z\to -2^+} g(z),\quad \lim_{z\to -2} g(z)$$

(b) *For* $g(x)=\frac{x+8}{x^2-9}$, *find*

$$\lim_{x\to 3^-} g(x),\quad \lim_{x\to 3^+} g(x),\quad \lim_{x\to 3} g(x)$$

(c) *For* $h(x)=\ln(-2x)$, *find*

$$\lim_{x\to 0^-} h(x),\quad \lim_{x\to 0^+} h(x),\quad \lim_{x\to 0} h(x)$$

(d) *For* $R(y) = \tan(2y)$, *find*

$$\lim_{y \to \frac{\pi}{2}^-} R(y), \quad \lim_{y \to \frac{\pi}{2}^+} R(y), \quad \lim_{y \to \frac{\pi}{2}} R(y)$$

Exercise 9 *Evaluate the following limits at infinity:*

(a) $\lim_{x \to \infty} f(x), \quad f(x) = \dfrac{4x^6 - 5x^2 + 1}{6 - 8x^2}$

(b) $\lim_{x \to -\infty} f(x), \quad f(x) = \dfrac{15x^4 - 9x^3}{3x^2 + 7x^4 + x}$

(c) $\lim_{x \to \infty} f(x), \quad f(x) = \dfrac{x^4 - 3x + 12}{5 - 2x^5}$

(d) $\lim_{x \to -\infty} f(x), \quad f(x) = \dfrac{x^5 - x^3 + 2}{6x^5 + 5x^2 + 1}$

(e) $\lim_{x \to \infty} f(x), \quad f(x) = \dfrac{\sqrt{5 + 4x^2}}{3 - x}$

(f) $\lim_{x \to -\infty} f(x), \quad f(x) = \dfrac{x + 5}{\sqrt{3x^2 + 2}}$

(g) $\lim_{x \to \infty} f(x), \quad f(x) = \dfrac{7 + 2x - 3x^2}{\sqrt{4 + x^2 + 6x^4}}$

Exercise 10 *Evaluate the following limits involving exponentials:*

(a) $\lim_{x \to -\infty} f(x), \quad f(x) = e^{5x^2 + x^4 + 2x}$

(b) $\lim_{x \to \infty} f(x), \quad f(x) = e^{5x^2 + x^4 + 2x}$

(c) $\lim_{x \to \infty} f(x), \quad f(x) = 3e^{5x} - e^{-6x} - 8e^{3x}$

(d) $\lim_{x \to -\infty} f(x), \quad f(x) = 4e^{-x} - 6e^{-4x} - e^{8x}$

(e) $\lim_{x \to \infty} f(x), \quad f(x) = \dfrac{e^{-2x} - 3e^{7x}}{8e^{7x} - 5e^{-2x}}$

2.10 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
4	Chapter 2.1–2.4 — Introduction to Calculus and Limits	https://youtu.be/saCGA-bm01o	
5	Chapter 2.5–2.6 — Infinite Limits and Limit to Infinity	https://youtu.be/D7LToTSwNU8	
6	Chapter 2.7 — Limit Formula and Examples	https://youtu.be/CaY3MVBYWRk	
7	Chapter 2.8 — Limit Solutions to Problem 6–19	https://youtu.be/3mbWTr3XspE	

Chapter 3

Continuity

3.1 Continuity of Functions

A function $f(x)$ is **continuous at a point** a if the following three conditions hold:

1. $f(a)$ is defined.
2. $\lim_{x\to a} f(x)$ exists.
3. $\lim_{x\to a} f(x) = f(a)$.

If any condition fails, f is **discontinuous** at a.

Procedure to Check Continuity

1. Verify that $f(a)$ exists.
2. Compute $\lim_{x\to a} f(x)$, including left- and right-hand limits if needed.
3. Compare $f(a)$ with the limit. Equality confirms continuity.

Example 75:

Continuity at a Point Determine if $f(x) = \frac{x^2-1}{x-1}$ is continuous at $x = 1$.
Solution: $f(1)$ is undefined since denominator $x - 1 = 0$.
Conclusion: f is discontinuous at $x = 1$ (removable discontinuity).

Example 76:

Continuity at a Point Determine if

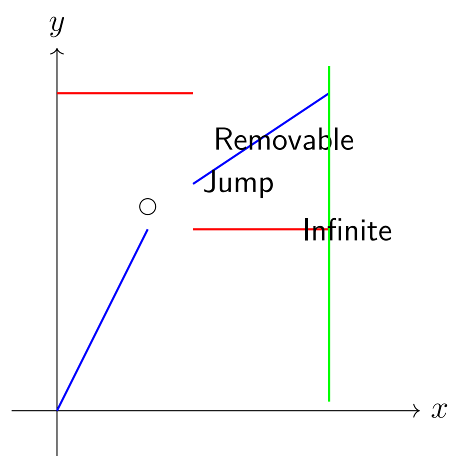
$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 5, & x = 2 \end{cases}$$

is continuous at $x = 2$.
Solution: $\lim_{x\to 2} f(x) = \lim_{x\to 2} x^2 = 4$. $f(2) = 5 \neq 4$.
Conclusion: f is discontinuous at $x = 2$ (removable discontinuity).

Types of Discontinuities

- **Removable:** Limit exists but $f(a) \neq \lim_{x\to a} f(x)$.
- **Jump:** $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$.
- **Infinite:** $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$.

Graphical Representation



Continuity over an Interval

- **Continuous from right at a:** $\lim_{x\to a^+} f(x) = f(a)$
- **Continuous from left at a:** $\lim_{x\to a^-} f(x) = f(a)$
- **Continuous on (a, b):** Continuous at every point in (a, b)
- **Continuous on $[a, b]$:** Continuous on (a, b), right-continuous at a, left-continuous at b

Example 77:

Continuity on Interval Find intervals of continuity for $f(x) = \frac{x+2}{x^2-1}$.
Solution: Denominator zero when $x^2 - 1 = 0 \implies x = \pm 1$.
Answer: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Example 78:

Continuity on Interval Find intervals of continuity for $f(x) = \sqrt{9 - x^2}$.
Solution: Inside square root $\geq 0 \implies 9 - x^2 \geq 0 \implies -3 \leq x \leq 3$.
Answer: $[-3, 3]$

Composite Functions and Trigonometric Continuity

If $f(x)$ is continuous at L and $\lim_{x\to a} g(x) = L$, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L)$$

Example 79:

Limit of a Composite Function Evaluate $\lim_{x\to 0} \sin(3x)$.
Solution: Let $g(x) = 3x$, $\lim_{x\to 0} g(x) = 0$, and $\sin$ is continuous at 0.

$$\lim_{x\to 0} \sin(3x) = \sin(0) = 0$$

$\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$ are continuous on their domains.

Intermediate Value Theorem (IVT)

If f is continuous on $[a, b]$ and z lies between $f(a)$ and $f(b)$, then $\exists c \in (a, b)$ such that $f(c) = z$.

Example 80:

Application of IVT Show that $f(x) = x^3 - x - 2$ has a root in $[1, 2]$.
Solution:

$$f(1) = 1 - 1 - 2 = -2 < 0, \quad f(2) = 8 - 2 - 2 = 4 > 0$$

f is continuous (polynomial). By IVT, $\exists c \in (1, 2)$ with $f(c) = 0$.

Example 81:

Root Existence Using IVT Show that $f(x) = \cos x - x$ has a root in $[0, 1]$.
Solution:

$$f(0) = 1 - 0 = 1 > 0, \quad f(1) = \cos 1 - 1 \approx 0.5403 - 1 = -0.4597 < 0$$

f is continuous (trig function). IVT guarantees $\exists c \in (0, 1)$ with $f(c) = 0$.

3.2 Problems

Problem 20 *Determine whether the function* $f(x) = \frac{x^2-9}{x-3}$ *is continuous at* $x = 3$.

Problem 21 *Examine the continuity of*

$$f(x) = \begin{cases} 2x + 1, & x < 1 \\ x^2 + 1, & x \geq 1 \end{cases}$$

at $x = 1$.

Problem 22 *Find the intervals of continuity for* $f(x) = \frac{1}{x^2-4}$.

Problem 23 *Classify the type of discontinuity of*

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 5, & x = 2 \end{cases}$$

at $x = 2$.

Problem 24 *Show that* $f(x) = x - \sin x$ *has a root in* $[0, 2]$ *using the Intermediate Value Theorem.*

Problem 25 *Determine whether* $f(x) = \cos(2x)$ *is continuous at* $x = \pi/4$.

Problem 26 *Evaluate* $\lim_{x\to 1} \sqrt{x^2 + 3} - 2$ *using the composite function theorem.*

Problem 27 *Check the continuity of*

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$$

at $x = 1$.

Problem 28 *Classify the discontinuity of* $f(x) = \tan x$ *at* $x = \pi/2$.

Problem 29 *Determine whether the function* $f(x) = \frac{x^3-8}{x-2}$ *is continuous at* $x = 2$.

3.3 Try it Yourself

Exercise 11 *Determine whether* $f(x) = \frac{x^2-4x+3}{x-1}$ *is continuous at* $x = 1$.

Exercise 12 *Examine the continuity of*

$$f(x) = \begin{cases} x^2 + 2x, & x < 0 \\ 3x + 1, & x \geq 0 \end{cases}$$

at $x = 0$.

Exercise 13 *Find the intervals of continuity for* $f(x) = \frac{x+1}{x^2-1}$.

Exercise 14 *Classify the type of discontinuity of*

$$f(x) = \begin{cases} \sin x, & x \neq \pi \\ 1, & x = \pi \end{cases}$$

at $x = \pi$.

Exercise 15 *Show that* $f(x) = x^3 - 3x + 1$ *has at least one root in* $[0, 2]$ *using the Intermediate Value Theorem.*

Exercise 16 *Determine whether* $f(x) = \sin(5x)$ *is continuous at* $x = \pi/10$.

Exercise 17 *Evaluate* $\lim_{x\to 2} \sqrt{3x+1} - \sqrt{7}$ *using the composite function theorem.*

Exercise 18 *Check the continuity of*

$$f(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ 6, & x = 3 \end{cases}$$

at $x = 3$.

Exercise 19 *Classify the discontinuity of* $f(x) = \sec x$ *at* $x = \pi/2$.

Exercise 20 *Determine whether* $f(x) = \frac{x^3-1}{x-1}$ *is continuous at* $x = 1$.

3.4 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
8	Chapter 3.1 — Introduction to Continuity and Examples	https://youtu.be/54gQJQ4dcu8	
9	Chapter 3.2 — Continuity – Solutions to Problem 20–29	https://youtu.be/XxFsEutFMv0	

Chapter 4

Differentiability

4.1 Derivatives

The **derivative** of a function $f(x)$ at a point $x = a$ measures the **instantaneous rate of change** of the function at that point. It is defined as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if the limit exists.
Geometrically, the derivative represents the **slope of the tangent line** to the curve $y = f(x)$ at $x = a$.

Interpretation of Derivative

- If $f'(a) > 0$, the function is increasing at $x = a$.
- If $f'(a) < 0$, the function is decreasing at $x = a$.
- If $f'(a) = 0$, the function may have a local maximum, minimum, or horizontal inflection at $x = a$.

Examples of Derivatives

Example 82:

Derivative of a Polynomial Find the derivative of $f(x) = 3x^3 - 5x^2 + 2x - 7$.
Solution: Using the power rule $\frac{d}{dx}[x^n] = nx^{n-1}$:

$$f'(x) = 9x^2 - 10x + 2$$

Example 83:

Derivative of a Trigonometric Function Find the derivative of $f(x) = \sin x + \cos x$.
Solution: Using standard derivatives $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$:

$$f'(x) = \cos x - \sin x$$

Example 84:

Derivative of an Exponential Function Find the derivative of $f(x) = e^{2x}$.
Solution: Using $\frac{d}{dx}e^u = e^u\frac{du}{dx}$, here $u = 2x$, $\frac{du}{dx} = 2$:

$$f'(x) = 2e^{2x}$$

Example 85:

Derivative of a Rational Function Find the derivative of $f(x) = \frac{1}{x^2}$.
Solution: Rewrite $f(x) = x^{-2}$, then use the power rule:

$$f'(x) = -2x^{-3} = -\frac{2}{x^3}$$

Example 86:

Derivative Using the Definition Find the derivative of $f(x) = x^2$ at $x = 3$ using the definition:

$$f'(a) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$

Solution: Let $a = 3$:

$$f'(3) = \lim_{h\to 0}\frac{(3+h)^2 - 9}{h} = \lim_{h\to 0}\frac{9 + 6h + h^2 - 9}{h} = \lim_{h\to 0}\frac{6h + h^2}{h} = \lim_{h\to 0}(6 + h) = 6$$

Differentiability

A function $f(x)$ is said to be **differentiable at a point** $x = a$ if its derivative exists at that point:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists.}$$

Key Points:

- If f is differentiable at $x = a$, then it is **continuous** at $x = a$.
- The converse is not necessarily true: continuity at $x = a$ does **not guarantee differentiability**.
- Geometrically, differentiability means the function has a **well-defined tangent** at $x = a$.

Geometric Interpretation

- The derivative $f'(a)$ represents the **slope of the tangent line** at $x = a$.
- If the curve has a sharp corner or cusp at $x = a$, it is **not differentiable**.
- If the curve has a vertical tangent, $f'(a)$ does not exist, so the function is **not differentiable** there.

Checking Differentiability at a Point

To check if $f(x)$ is differentiable at $x = a$:

1. Check if $f(x)$ is **continuous** at $x = a$.
2. Compute the **left-hand derivative**:

$$f'_{-}(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

3. Compute the **right-hand derivative**:

$$f'_{+}(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

4. If $f'_{-}(a) = f'_{+}(a)$ (finite), then f is differentiable at $x = a$. Otherwise, it is not.

Example 87:

Continuity: $|x|$ is continuous at 0.

Left derivative:

$$f'_{-}(0) = \lim_{h \to 0^-} \frac{|h|}{h} = -1$$

Right derivative:

$$f'_{+}(0) = \lim_{h \to 0^+} \frac{|h|}{h} = 1$$

Since $f'_{-}(0) \neq f'_{+}(0)$, $f(x)$ is **not differentiable** at 0.

Example 88:

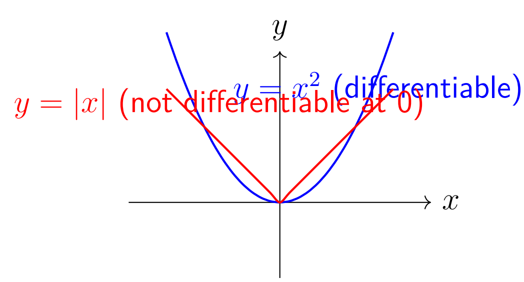
Continuity: x^2 is continuous everywhere.
Both derivatives:

$$f'(2) = \lim_{h \to 0} \frac{(2+h)^2 - 4}{h} = \lim_{h \to 0}(4+h) = 4$$

Thus, $f(x)$ is **differentiable** at $x = 2$ with slope 4.

Visualizing Differentiability

Curves Illustration



Example 89:

Continuity: $\sqrt{|x|}$ is continuous at 0.
Left derivative:

$$f'_{-}(0) = \lim_{h \to 0^-} \frac{\sqrt{-h}}{h} = -\infty$$

Right derivative:

$$f'_{+}(0) = \lim_{h \to 0^+} \frac{\sqrt{h}}{h} = +\infty$$

Since derivatives diverge, $f(x)$ is **not differentiable** at 0.

Example 90:

$$f(x) = \begin{cases} x^2, & x \leq 1 \\ 2x - 1, & x > 1 \end{cases}$$

Continuity at $x = 1$: From left: $f(1^-) = 1$, from right: $f(1^+) = 1$. Continuous.
Left derivative:

$$f'_-(1) = \lim_{h \to 0^-} \frac{(1+h)^2 - 1}{h} = 2$$

Right derivative:

$$f'_+(1) = \lim_{h \to 0^+} \frac{(2(1+h) - 1) - 1}{h} = 2$$

Since $f'_-(1) = f'_+(1) = 2$, f is **differentiable at** $x = 1$.

Example 91:

Differentiable Trigonometric Function Consider $f(x) = \sin x$.
Solution: $f'(x) = \cos x$, which exists for all $x \in \mathbb{R}$.
Conclusion: Differentiable everywhere.

4.2 Problems

Problem 30 *Determine whether* $f(x) = x^3 - 3x + 1$ *is differentiable at* $x = 1$.

Problem 31 *Examine the differentiability of* $f(x) = |x - 2|$ *at* $x = 2$.

Problem 32 *Check differentiability of the piecewise function:*

$$f(x) = \begin{cases} x^2, & x \leq 0 \\ x, & x > 0 \end{cases}$$

at $x = 0$.

Problem 33 *Find all points where* $f(x) = \sqrt[3]{x}$ *is not differentiable.*

Problem 34 *Determine whether* $f(x) = \sin x + \cos x$ *is differentiable at* $x = \pi/4$.

Problem 35 *Check differentiability of* $f(x) = x|x|$ *at* $x = 0$.

Problem 36 *Determine whether* $f(x) = e^x \sin x$ *is differentiable at* $x = 0$.

Problem 37 *Check differentiability of*

$$f(x) = \begin{cases} x^2 + 2x, & x \leq 1 \\ 3x - 1, & x > 1 \end{cases}$$

at $x = 1$.

Problem 38 *Determine the differentiability of* $f(x) = \tan x$ *at* $x = \pi/2$.

Problem 39 *Find the derivative of* $f(x) = x^2 \sin x$ *and check its differentiability at* $x = 0$.

4.3 Try it Yourself

Exercise 21 *Check whether* $f(x) = |x+1| + |x-1|$ *is differentiable at* $x = -1$ *and* $x = 1$.

Exercise 22 *Determine the points of non-differentiability of* $f(x) = x^{2/3}(x-2)^{1/3}$.

Exercise 23 *Examine differentiability of the piecewise function:*

$$f(x) = \begin{cases} x^2 + 1, & x < 2 \\ 4x - 3, & x \geq 2 \end{cases}$$

at $x = 2$.

Exercise 24 *Find* a *such that the function*

$$f(x) = \begin{cases} ax^2 + 2x, & x \leq 1 \\ x^2 + 3, & x > 1 \end{cases}$$

is differentiable at $x = 1$.

Exercise 25 *Check differentiability of* $f(x) = \tan^{-1}(x^2 - 1)$ *at* $x = 1$.

Exercise 26 *Determine whether* $f(x) = x^2 |x-1|$ *is differentiable at* $x = 1$.

Exercise 27 *Check the differentiability of* $f(x) = \sqrt{x^2 + 4x + 3}$ *at* $x = -1$ *and* $x = -3$.

Exercise 28 *Determine the differentiability of* $f(x) = e^{|x|}$ *at* $x = 0$.

Exercise 29 *Examine differentiability of* $f(x) = \sin(x)|x|$ *at* $x = 0$.

Exercise 30 *Find all points where* $f(x) = x^{1/3} + |x|$ *is not differentiable.*

4.4 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
10	Chapter 4.1 — Introduction to Differentiability and Examples	https://youtu.be/ZG4LQ0I7iVw	
11	Chapter 4.2 — Differentiability – Solutions to Problem 30–39	https://youtu.be/iqr-e1rSjUQ	

Chapter 5

Optimization (Maxima/Minima)

5.1 Rate of Change

Derivatives as Rates of Change

Introduction

One of the fundamental interpretations of the derivative is that it measures the **rate of change** of a function. Specifically, if $f(x)$ represents a quantity that depends on x, then the derivative $f'(x)$ tells us how rapidly $f(x)$ is changing with respect to x at any given point.
This idea is widely applicable in physics, engineering, and economics, for example, in measuring velocity as the rate of change of position, growth rates in populations, or marginal cost in economics.

Examples of Rate of Change

Example 92:

Velocity of a Moving Particle The position of a particle is given by $s(t) = 5t^2 + 2t$. Find its velocity at $t = 3$ seconds.
Solution: Velocity is the derivative of position:

$$v(t) = s'(t) = \frac{d}{dt}(5t^2 + 2t) = 10t + 2$$

At $t = 3$:

$$v(3) = 10(3) + 2 = 32 \text{ units/sec}$$

Example 93:

Marginal Cost The cost (in dollars) of producing x units of a product is $C(x) = 50 + 4x + 0.1x^2$. Find the marginal cost when $x = 20$ units.
Solution: Marginal cost is the derivative of the total cost:

$$C'(x) = \frac{d}{dx}(50 + 4x + 0.1x^2) = 4 + 0.2x$$

At $x = 20$:

$$C'(20) = 4 + 0.2(20) = 4 + 4 = 8 \text{ dollars/unit}$$

Example 94:

Rate of Change of Temperature The temperature of a cup of coffee at time t minutes is modeled by $T(t) = 80 - 5\ln(t+1)$. Find the rate at which the temperature is decreasing at $t = 4$ minutes.
Solution: Rate of change of temperature:

$$T'(t) = \frac{d}{dt}(80 - 5\ln(t+1)) = -\frac{5}{t+1}$$

At $t = 4$:

$$T'(4) = -\frac{5}{5} = -1 \text{ degree/minute}$$

Conclusion: The coffee is cooling at 1 degree per minute at $t = 4$.

Example 95:

Rate of Change of Area The side of a square grows with time according to $s(t) = 3t + 2$ meters. Find the rate at which the area of the square changes at $t = 2$ seconds.
Solution: Area $A = s^2$, so

$$\frac{dA}{dt} = 2s \cdot \frac{ds}{dt} = 2(3t + 2) \cdot 3 = 6(3t + 2)$$

At $t = 2$:

$$\frac{dA}{dt} = 6(6 + 2) = 6 \cdot 8 = 48 \text{ m}^2\text{/sec}$$

The derivative is a versatile tool to calculate **instantaneous rates of change** in various contexts: motion, economics, population dynamics, temperature, and geometric growth. Recognizing the connection between the function and its derivative allows us to model real-world phenomena effectively.

5.2 Critical Points

Let $f(x)$ be a function defined on some interval. A point $x = c$ is called a **critical point** of $f(x)$ if:

- $f(c)$ exists, and
- either $f'(c) = 0$ **or** $f'(c)$ does not exist.

Key Idea

Critical points indicate where a function may have:

- a local maximum,
- a local minimum,
- or a **saddle point** (point of inflection where derivative is 0 but no extremum).

Note: Finding critical points is only the first step. To classify them as max/min/saddle, one usually applies the **First Derivative Test** or the **Second Derivative Test**.

Examples

Example 96: Polynomial Function

Determine all critical points of

$$f(x) = 4x^3 - 12x^2 + 7x + 5$$

Solution: Derivative:

$$f'(x) = 12x^2 - 24x + 7$$

Set $f'(x) = 0$:

$$12x^2 - 24x + 7 = 0 \implies x = 1 \pm \frac{\sqrt{15}}{6}$$

Conclusion: Critical points are $x = 1 \pm \frac{\sqrt{15}}{6}$.

Example 97: Root Function

Find the critical points of

$$g(t) = \sqrt{t}(t - 2), \quad t \geq 0$$

Solution: Derivative:

$$g'(t) = \frac{1}{2\sqrt{t}}(t - 2) + \sqrt{t} = \frac{3t - 2}{2\sqrt{t}}$$

Set $g'(t) = 0$:

$$3t - 2 = 0 \implies t = \frac{2}{3}$$

Also, $g'(t)$ undefined at $t = 0$ (but $g(0) = 0$ exists).
Conclusion: Critical points are $t = 0$ and $t = \frac{2}{3}$.

Example 98: Trigonometric Function

Find the critical points of

$$y = 5x - 3\sin(2x)$$

Solution: Derivative:

$$y' = 5 - 6\cos(2x)$$

Set $y' = 0$:

$$\cos(2x) = \frac{5}{6}$$

Conclusion: Infinite critical points:

$$x = \frac{1}{2}\cos^{-1}\left(\frac{5}{6}\right) + n\pi, \quad n \in \mathbb{Z}$$

Example 99: Exponential Function

Determine all critical points of

$$h(t) = te^{-t^2}$$

Solution: Derivative:

$$h'(t) = e^{-t^2}(1 - 2t^2)$$

Set $h'(t) = 0$:

$$1 - 2t^2 = 0 \implies t = \pm\frac{1}{\sqrt{2}}$$

Conclusion: Critical points at $t = \pm\frac{1}{\sqrt{2}}$.

Example 100: Logarithmic Function

Find critical points of

$$f(x) = x^2\ln(x), \quad x > 0$$

Solution: Derivative:

$$f'(x) = 2x\ln x + x = x(2\ln x + 1)$$

Set $f'(x) = 0$: $x = 0$ (not in domain), or $\ln x = -\frac{1}{2} \implies x = e^{-1/2}$.
Conclusion: Critical point at $x = e^{-1/2}$.

Example 101: Product of Polynomial and Exponential

Determine all critical points of

$$f(x) = xe^{x^2}$$

Solution: Derivative:

$$f'(x) = e^{x^2}(1 + 2x^2)$$

Since $e^{x^2} > 0$ for all x, we need $1 + 2x^2 = 0$, which has no real solution.
Conclusion: No critical points in the real domain.

Critical points arise when:

- $f'(x) = 0$ (possible max, min, or saddle point),
- $f'(x)$ does not exist (but $f(x)$ is defined).

Important:

- Critical points do not guarantee maxima or minima.
- Example: $f(x) = x^3$ has a critical point at $x = 0$ which is a **saddle point**.
- To classify, use the **First Derivative Test** or **Second Derivative Test**.

5.3 Minima and Maxima

Maximum and Minimum Values of a Function

Let $f(x)$ be a function defined on a domain D.

- $f(x)$ has an **absolute (global) maximum** at $x = c$ if

$$f(c) \geq f(x) \quad \forall x \in D$$

- $f(x)$ has a **absolute (global) minimum** at $x = c$ if

$$f(c) \leq f(x) \quad \forall x \in D$$

- $f(x)$ has a **relative (local) maximum** at $x = c$ if $f(c) \geq f(x)$ for all x in some open interval around c.
- $f(x)$ has a **relative (local) minimum** at $x = c$ if $f(c) \leq f(x)$ for all x in some open interval around c.

Notes:

- Relative extrema occur **interior to the domain**, not at endpoints.
- Absolute extrema may occur at endpoints or at relative extrema inside the domain.

Extrema

The **extrema** of a function refer collectively to all maximum and minimum values:

- **Relative extrema** = local maximums and minimums
- **Absolute extrema** = global maximums and minimums

Examples

Example 102:

Quadratic Function on a Closed Interval Find the absolute and relative extrema of

$$f(x) = x^2 - 4x + 3 \quad \text{on } [0, 3]$$

Solution:
Derivative: $f'(x) = 2x - 4$. Set $f'(x) = 0 \implies 2x - 4 = 0 \implies x = 2$ (critical point).
Check endpoints and critical point:

$$f(0) = 3, \quad f(2) = -1, \quad f(3) = 0$$

Conclusion: Absolute maximum $= f(0) = 3$, Absolute minimum $= f(2) = -1$, Relative minimum $= f(2) = -1$, Relative maximum $= f(0) = 3$ (endpoint, only absolute).

Example 103:

Cubic Function on a Closed Interval Find absolute and relative extrema of

$$f(x) = x^3 - 3x^2 + 2 \quad \text{on } [-1, 2]$$

Solution:
Derivative: $f'(x) = 3x^2 - 6x = 3x(x - 2)$. Critical points: $x = 0, 2$
Check endpoints and critical points:

$$f(-1) = -2, \quad f(0) = 2, \quad f(2) = -2$$

Conclusion: Absolute maximum $= f(0) = 2$, Absolute minimum $= f(-1) = -2$ (also occurs at $x = 2$), Relative maximum $= f(0) = 2$, Relative minimum = none in interior.

Example 104:

Trigonometric Function Find extrema of

$$f(x) = \sin x \quad \text{on } [0, 2\pi]$$

Solution:
Derivative: $f'(x) = \cos x$. Set $f'(x) = 0 \implies \cos x = 0 \implies x = \frac{\pi}{2}, \frac{3\pi}{2}$
Check endpoints and critical points:

$$f(0) = 0, f(2\pi) = 0, f(\pi/2) = 1, f(3\pi/2) = -1$$

Conclusion: Absolute maximum = 1 at $x = \pi/2$, Absolute minimum = -1 at $x = 3\pi/2$, Relative maximum = 1 at $x = \pi/2$, Relative minimum = -1 at $x = 3\pi/2$.

Example 105:

Function with No Relative Extrema

$$f(x) = x^3 \quad \text{on } [-2, 2]$$

Solution:
Derivative: $f'(x) = 3x^2 = 0 \implies x = 0$
Check second derivative or graph: $f''(x) = 6x$ At $x = 0$, $f''(0) = 0 \rightarrow$ inflection point, not maximum or minimum
Check endpoints for absolute extrema: $f(-2) = -8, f(2) = 8$
Conclusion: Absolute maximum = 8 at $x = 2$, Absolute minimum = -8 at $x = -2$, No relative extrema.

Important Theorems

Extreme Value Theorem

If $f(x)$ is continuous on a closed interval $[a, b]$, then there exist $c, d \in [a, b]$ such that $f(c)$ is an absolute maximum and $f(d)$ is an absolute minimum.

Fermat's Theorem

If $f(x)$ has a relative extrema at $x = c$ and $f'(c)$ exists, then $x = c$ is a critical point of $f(x)$, i.e., $f'(c) = 0$.

- Not all critical points are relative extrema (e.g., $f(x) = x^3$ at $x = 0$).
- Relative extrema can also occur at critical points where $f'(c)$ does not exist (e.g., $f(x) = |x|$ at $x = 0$).
- Absolute extrema may or may not occur at critical points; endpoints must also be checked.

5.4 Absolute Extrema

Finding Absolute Extrema on a Closed Interval

When a function $f(x)$ is continuous on a **closed interval** $[a, b]$, the **Extreme Value Theorem** guarantees that $f(x)$ attains both an absolute maximum and an absolute minimum somewhere in the interval. These extrema can occur either at the **endpoints** or at the **critical points** of the function.

Procedure for Finding Absolute Extrema

To find the absolute extrema of $f(x)$ on $[a, b]$:

1. Verify that $f(x)$ is continuous on $[a, b]$.

2. Find all **critical points** c in (a, b) where $f'(c) = 0$ or $f'(c)$ does not exist.
3. Evaluate $f(x)$ at all critical points and at the endpoints $x = a$ and $x = b$.
4. Identify the largest and smallest values; these are the **absolute maximum** and **absolute minimum**, respectively.

Geometric Interpretation

- Only critical points inside the interval matter; critical points outside the interval should be ignored.
- Absolute extrema can occur at either critical points or endpoints.
- A small change in the interval may change which points are the absolute extrema.

Examples

Example 106:

Polynomial Function on a Closed Interval Find the absolute extrema of

$$f(x) = x^3 - 6x^2 + 9x + 1 \quad \text{on } [0, 4]$$

Solution:
Derivative: $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$
Critical points in $[0, 4]$: $x = 1, 3$
Evaluate $f(x)$ at endpoints and critical points:

$$f(0) = 1, \quad f(1) = 5, \quad f(3) = 1, \quad f(4) = 1$$

Conclusion: Absolute maximum $= f(1) = 5$, Absolute minimum $= f(0) = f(3) = f(4) = 1$

Example 107:

Cubic Function with Restricted Interval Find the absolute extrema of

$$f(x) = 2x^3 - 9x^2 + 12x + 5 \quad \text{on } [1, 3]$$

Solution:
Derivative: $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$
Critical points in $[1, 3]$: $x = 1, 2$ (ignore $x = 1$ if already endpoint)
Evaluate $f(x)$ at endpoints and critical points:

$$f(1) = 10, \quad f(2) = 8, \quad f(3) = 11$$

Conclusion: Absolute maximum $= f(3) = 11$, Absolute minimum $= f(2) = 8$

Example 108:

Trigonometric Function Determine the minimum and maximum values of

$$f(t) = \sin(2t) + t \quad \text{on } [0, \pi]$$

Solution:
Derivative: $f'(t) = 2\cos(2t) + 1$
Set $f'(t) = 0 \implies 2\cos(2t) + 1 = 0 \implies \cos(2t) = -\frac{1}{2} \implies 2t = \frac{2\pi}{3}, \frac{4\pi}{3} \implies t = \frac{\pi}{3}, \frac{2\pi}{3}$
Evaluate at critical points and endpoints:

$$f(0) = 0, \quad f(\pi) = \pi, \quad f(\pi/3) = \frac{\sqrt{3}}{2} + \frac{\pi}{3}, \quad f(2\pi/3) = -\frac{\sqrt{3}}{2} + \frac{2\pi}{3}$$

Conclusion: Absolute maximum $= f(\pi) = \pi$, Absolute minimum $= f(0) = 0$

Example 109:

Exponential Function Find the extrema of

$$f(x) = xe^{-x^2} \quad \text{on } [-2, 2]$$

Solution:
Derivative: $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}(1 - 2x^2)$
Critical points: $1 - 2x^2 = 0 \implies x = \pm\frac{1}{\sqrt{2}}$
Evaluate at endpoints and critical points:

$$f(-2) \approx -0.036, \quad f(2) \approx 0.036, \quad f(\pm 1/\sqrt{2}) = \pm 0.43$$

Conclusion: Absolute maximum ≈ 0.43 at $x = 1/\sqrt{2}$, Absolute minimum ≈ -0.43 at $x = -1/\sqrt{2}$

Example 110:

Function with Nonexistent Derivative at Critical Point Determine extrema of

$$f(x) = |x - 1| \quad \text{on } [0, 3]$$

Solution:
Derivative does not exist at $x = 1$ (critical point).
Evaluate endpoints and critical point:

$$f(0) = 1, \quad f(1) = 0, \quad f(3) = 2$$

Conclusion: Absolute maximum $= f(3) = 2$, Absolute minimum $= f(1) = 0$

5.5 Increasing/Decreasing and Concavity/Convexity

Increasing and Decreasing Functions

- A function $f(x)$ is **increasing** on an interval if $f'(x) > 0$ for all x in that interval.
- A function $f(x)$ is **decreasing** on an interval if $f'(x) < 0$ for all x in that interval.
Procedure: 1. Compute $f'(x)$. 2. Solve $f'(x) = 0$ to find critical points. 3. Test intervals between critical points to determine sign of $f'(x)$.

Concavity and Convexity

- A function $f(x)$ is **concave up** (convex) if $f''(x) > 0$. - A function $f(x)$ is **concave down** if $f''(x) < 0$. - **Inflection points** occur where $f''(x) = 0$ and concavity changes.
Procedure: 1. Compute $f''(x)$. 2. Solve $f''(x) = 0$. 3. Test intervals to see where $f''(x)$ is positive or negative.

Example 111:

Let $f(x) = x^3 - 3x^2 + 2$.
Compute derivative: $f'(x) = 3x^2 - 6x = 3x(x-2)$.
Critical points: $x = 0, 2$.
Test intervals: - $(-\infty, 0)$: $f'(x) > 0 \to$ increasing - $(0, 2)$: $f'(x) < 0 \to$ decreasing - $(2, \infty)$: $f'(x) > 0 \to$ increasing

Example 112:

Let $f(x) = x^3 - 3x^2 + 2$.
Second derivative: $f''(x) = 6x - 6$.
Set $f''(x) = 0 \implies x = 1$.
Test intervals: - $(-\infty, 1)$: $f''(x) < 0 \to$ concave down - $(1, \infty)$: $f''(x) > 0 \to$ concave up.
Inflection point at $x = 1$.

5.6 Mean Value Theorem

Rolle's Theorem

Theorem (Rolle's Theorem): Let $f(x)$ be a function satisfying:

1. $f(x)$ is continuous on the closed interval $[a, b]$,
2. $f(x)$ is differentiable on the open interval (a, b),
3. $f(a) = f(b)$.

Then there exists at least one number $c \in (a, b)$ such that

$$f'(c) = 0$$

or equivalently, $f(x)$ has a critical point in (a, b).

Example 113:

Application of Rolle's Theorem Show that $f(x) = x^3 - 3x + 2$ has at least one critical point in $[-2, 2]$.

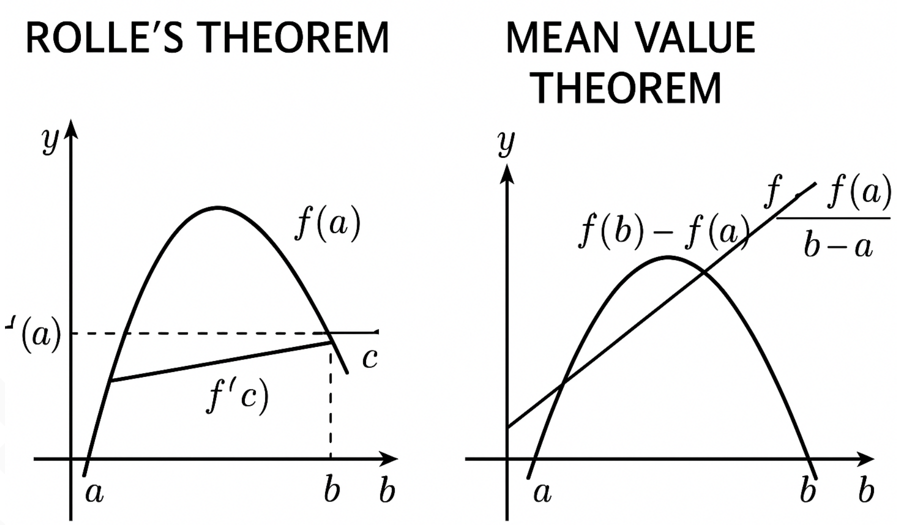
solution:

Check the conditions:

- $f(x)$ is a polynomial $\implies$ continuous and differentiable everywhere.
- $f(-2) = (-8) + 6 + 2 = 0$, $f(2) = 8 - 6 + 2 = 4$. Since $f(-2) \neq f(2)$, Rolle's Theorem cannot be applied here directly.

Instead, if we pick interval $[0, 1]$, $f(0) = 2$, $f(1) = 0$. Then we can adjust and check other intervals for equal values. The key point: Rolle's Theorem ensures that whenever $f(a) = f(b)$, there exists a c with $f'(c) = 0$.

Mean Value Theorem (MVT)



Theorem (MVT): If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists at least one $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Geometric Interpretation

The Mean Value Theorem states that there exists at least one point c where the tangent to the curve is **parallel to the secant line** connecting $(a, f(a))$ and $(b, f(b))$.

Example 114:

Using MVT Find all c that satisfy the conclusion of MVT for

$$f(x) = x^3 - x \quad \text{on } [0, 2].$$

Solution:
Slope of secant:

$$\frac{f(2) - f(0)}{2 - 0} = \frac{6 - 0}{2} = 3$$

Derivative: $f'(x) = 3x^2 - 1$
Set $f'(c) = 3 \implies 3c^2 - 1 = 3 \implies 3c^2 = 4 \implies c = \frac{2}{\sqrt{3}} \approx 1.155$

Example 115:

Bounding Function Values Using MVT Suppose $f(x)$ is continuous and differentiable on $[1, 5]$, with $f'(x) \leq 4$ and $f(1) = 3$. What is the largest possible value of $f(5)$?
Solution:
By MVT, for some $c \in (1, 5)$:

$$f'(c) = \frac{f(5) - f(1)}{5 - 1} \leq 4 \implies \frac{f(5) - 3}{4} \leq 4 \implies f(5) \leq 19$$

Conclusion: Maximum possible $f(5) = 19$

Applications of MVT

If $f'(x) = 0$ for all x in (a, b), then $f(x)$ is constant on (a, b).
Proof: For any $x_1, x_2 \in (a, b)$, by MVT, $\exists c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \implies f(x_2) = f(x_1)$$

If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + C$ for some constant C.

Proof

Define $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) = 0$ for all $x \in (a, b)$. By the previous fact, $h(x)$ is constant. Hence, $f(x) = g(x) + C$.

5.7 Problems

Problem 40 *Determine the critical points of*

$$f(x) = 6x^3 - 15x^2 + 8x - 1$$

Problem 41 *Find the absolute extrema of*

$$f(x) = 2x^3 - 9x^2 + 12x + 1$$

on $[-2, 3]$.

Problem 42 *Verify Rolle's Theorem for*

$$f(x) = x^2 - 6x + 8$$

on $[2, 4]$.

Problem 43 *Verify the Mean Value Theorem for*

$$f(x) = 3x^3 - 12x^2 + 7x - 1$$

on $[1, 3]$.

Problem 44 *Determine intervals of increase/decrease and concavity for* $f(x) = x^3 - 6x^2 + 9x + 1$.

Problem 45 *Determine extrema for* $f(x) = x^4 - 4x^3 + 6x^2$.

Problem 46 *Determine extrema for* $f(x) = 2x^3 - 9x^2 + 12x - 1$.

Problem 47 *Find intervals where* $f(x) = x^3 - 3x + 1$ *is concave up and concave down.*

5.8 Try it Yourself

Exercise 31 *Determine the critical points of*

$$f(x) = 2x^4 - 8x^3 + 6x^2 - x + 1$$

Exercise 32 *Find absolute extrema of*

$$f(x) = x^3 - 5x^2 + 6x + 2$$

on $[-1, 3]$.

Exercise 33 *Use L'Hospital's Rule to evaluate:*

$$\lim_{x \to 0} \frac{e^{3x} - 1 - 3x}{x^2}$$

Exercise 34 *Verify Rolle's Theorem for*

$$f(x) = x^2 - 5x + 6$$

on $[2, 3]$.

Exercise 35 *Use L'Hospital's Rule to evaluate:*

$$\lim_{x \to \infty} \frac{\ln(4x + 1)}{x}$$

Exercise 36 *Determine increasing/decreasing intervals and concavity for* $f(x) = x^3 - 4x^2 + 5x$.

Exercise 37 *Find critical points and concavity for* $f(x) = 2x^4 - 8x^3 + 6x^2$.

Exercise 38 *Determine intervals where* $f(x) = x^3 - 3x^2 + x + 2$ *is increasing/decreasing.*

Exercise 39 *Determine concavity and inflection points for* $f(x) = x^4 - 4x^2 + 1$.

Exercise 40 *Determine increasing/decreasing intervals and concavity for* $f(x) = x^3 - 6x^2 + 11x - 6$.

5.9 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
12	Chapter 5.1–5.2: Optimization – Rate of Change, Critical Points, and Examples	https://youtu.be/PjENjYthne0	
13	Chapter 5.3–5.4: Optimization – Relative/Local and Global/Absolute Maxima Minima (Extrema)	https://youtu.be/SKYr0E-ecvw	
14	Chapter 5.5–5.6: Optimization – Concavity, Convexity, Rolle's and Mean Value Theorem	https://youtu.be/ftoArL_RYtc	
15	Chapter 5.7: Optimization – Solutions to Problems 40–47	https://youtu.be/xc4m8MS5J88	

Chapter 6

Taylor series

6.1 Taylor Series Expansion

Taylor Series Definition

If a function $f(x)$ has derivatives of all orders at $x = a$, the Taylor Series about $x = a$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

If $a = 0$, the series is called a **Maclaurin Series**.

Common Maclaurin Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Taylor Polynomial and Remainder

The nth-degree Taylor polynomial:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Remainder (error) term:

$$R_n(x) = f(x) - T_n(x)$$

If $\lim_{n\to\infty} R_n(x) = 0$, the series converges to $f(x)$.

Examples

Example 116:

Find the Maclaurin series for $f(x) = e^{2x}$.
solution $f^{(n)}(x) = 2^n e^{2x}$, so $f^{(n)}(0) = 2^n$. Hence, the Maclaurin series:

$$f(x) = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n = 1 + 2x + 2^2 \frac{x^2}{2!} + 2^3 \frac{x^3}{3!} + \cdots$$

Example 117:

Find the Taylor series for $f(x) = \ln(1 + x)$ about $x = 0$.
solution $f'(x) = 1/(1 + x)$, $f''(x) = -1/(1 + x)^2$, $f^{(n)}(x) = (-1)^{n-1}(n - 1)!/(1 + x)^n$ So at $x = 0$: $f^{(n)}(0) = (-1)^{n-1}(n - 1)!$ Taylor series:

$$\ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Example 118:

Find the Maclaurin series for $f(x) = \sin(3x)$.
solution Use $\sin(kx)$:

$$\sin(3x) = \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{(2n + 1)!} = 3x - \frac{27x^3}{6} + \frac{243x^5}{120} - \cdots$$

6.2 Using Taylor Series for Limits

Using Taylor Series for Limits

Taylor or Maclaurin series can simplify limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by expanding functions near the point. This often avoids repeated differentiation in L'Hôpital's Rule.

Example 119:

Evaluate:

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$

Solution: Maclaurin series: $e^x = 1 + x + \frac{x^2}{2} + \cdots$

$$e^x - 1 - x = \frac{x^2}{2} + \cdots$$

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}$$

Example 120:

Evaluate:

$$\lim_{x \to 0} \frac{\ln(1+x) - x + \frac{x^2}{2}}{x^3}$$

Solution: Maclaurin series: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

$$\ln(1+x) - x + \frac{x^2}{2} = \frac{x^3}{3} + \cdots$$

$$\lim_{x \to 0} \frac{\ln(1+x) - x + \frac{x^2}{2}}{x^3} = \frac{1}{3}$$

Example 121:

Evaluate:

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}$$

Solution: Maclaurin series: $\tan x = x + \frac{x^3}{3} + \cdots$

$$\tan x - x = \frac{x^3}{3} + \cdots$$

$$\lim_{x \to 0} \frac{\tan x - x}{x^3} = \frac{1}{3}$$

6.3 Problems

Problem 48 *Find the Maclaurin series for* $f(x) = \cos(2x)$ *up to* x^6.

Problem 49 *Find the Taylor series of* $f(x) = e^{-x^2}$ *about* $x = 0$ *up to* x^4.

Problem 50 *Evaluate*

$$\lim_{x \to 0} \frac{\cos(x) - e^{-x^2/2}}{x^4}.$$

Problem 51 *Evaluate*

$$\lim_{x \to 0} \frac{e^{2x} - 1 - 2x - 2x^2}{x^7}.$$

Problem 52 *Compute*

$$\lim_{x \to 0} \frac{\ln(\cos x)}{x^2}.$$

6.4 Try it Yourself

Exercise 41 *Find the Maclaurin series for* $f(x) = \sin(4x)$ *up to* x^5.

Exercise 42 *Find the Taylor series for* $f(x) = e^{x^2}$ *about* $x = 0$ *up to* x^4.

Exercise 43 *Find the Maclaurin series for* $f(x) = \ln(1+x^2)$ *up to* x^4.

Exercise 44 *Find the Taylor series for* $f(x) = (1+x)^{-1}$ *about* $x = 0$ *up to* x^3.

Exercise 45 *Find the Taylor series for* $f(x) = \tan^{-1}(2x)$ *about* $x = 0$ *up to* x^5.

Exercise 46 *Evaluate*

$$\lim_{x \to 0} \frac{1 - \cos x - \frac{x^2}{2}}{x^4}.$$

Exercise 47 *Evaluate*

$$\lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}.$$

Exercise 48 *Evaluate*

$$\lim_{x \to 0} \frac{e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{6}}{x^4}.$$

6.5 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
16	Chapter 6.1–6.2 — Taylor and Maclaurin Series – Applications to Limits and Examples	https://youtu.be/U2bTdqOLI1g	
17	Chapter 6.3 — Taylor and Maclaurin Series – Solutions to Problems 48–52	https://youtu.be/q5n3QCe4ICQ	

Chapter 7

GATE PYQs

7.1 Questions

GATEPYQ 1 *Consider*

$$f(x) = \begin{cases} ax + b, & x < 1 \\ x^3 + x^2 + 1, & x \geq 1 \end{cases}$$

If f is differentiable everywhere, find b (rounded to one decimal place).

GATEPYQ 2 *Let f be continuous with*

$$f(x) = 1 - f(2 - x)$$

Then

$$\int_0^2 f(x)dx = ?$$

- *A: 0*
- *B: 1*
- *C: 2*
- *D: -1*

GATEPYQ 3 *Let $f(x) = x^3 + 15x^2 - 33x - 36$. Which statements are true?*

- *A: f has no local maximum*
- *B: f has a local maximum*
- *C: f has no local minimum*
- *D: f has a local minimum*

GATEPYQ 4 *Evaluate*

$$\lim_{x\to 0^+} \frac{\sqrt{x}}{1-e^{2\sqrt{x}}}$$

GATEPYQ 5 *If f is continuous on [-3,3], differentiable on (-3,3), with $f'(x) \leq 2$ and $f(-3) = 7$, then $f(3) \leq ?$*

GATEPYQ 6 *Compute*

$$\lim_{x\to -3} \frac{\sqrt{2x+22}-4}{x+3}$$

(Rounded to 2 decimal places)

GATEPYQ 7 *Which of the following functions is increasing everywhere on [0,1]?*

- *I: e^{-x}*
- *II: $x^2 - \sin x$*
- *III: $\sqrt{x^3+1}$*

Options:

- *A: III only*
- *B: II only*
- *C: II and III only*
- *D: I and III only*

GATEPYQ 8 *Compute*

$$\lim_{x\to 3} \frac{x^4-81}{2x^2-5x-3}$$

- *A: 1*
- *B: 53/12*
- *C: 108/7*
- *D: Does not exist*

GATEPYQ 9 *The domain of $\log(\log \sin x)$ is:*

- *A: $0 < x < \pi$*
- *B: $2n\pi < x < (2n+1)\pi$, $n \in \mathbb{Z}$*
- *C: Empty set*
- *D: None of the above*

GATEPYQ 10 *Given $f(x) = R\sin(\pi x/2) + S$, $f'(\frac{1}{2}) = \sqrt{2}$, $\int_0^1 f(x)dx = 2R/\pi$, find R, S.*

- *A: $2/\pi$ and $16/\pi$*

- *B:* $2/\pi$ *and 0*
- *C:* $4/\pi$ *and 0*
- *D:* $4/\pi$ *and* $16/\pi$

GATEPYQ 11 *Evaluate*

$$\lim_{x\to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$$

- *A: 0*
- *B: -1*
- *C: 1*
- *D: Does not exist*

GATEPYQ 12 *Evaluate*

$$\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$$

- *A: 0*
- *B: -1*
- *C: 1*
- *D: 1/2*

GATEPYQ 13 *If* $f(x)$ *is polynomial,* $g(x) = f'(x)$*, degree of* $f(x) + f(-x)$ *is 10, find degree of* $g(x) - g(-x)$.

GATEPYQ 14 *Evaluate*

$$\lim_{x\to 4} \frac{\sin(x-4)}{x-4}$$

GATEPYQ 15 *Evaluate*

$$\lim_{x\to\infty} (1+x^2)^{e^{-x}}$$

- *A: 0*
- *B: 1/2*
- *C: 1*
- *D:* ∞

GATEPYQ 16 *Let* $f(x) = x^{-1/3}$ *and* A *be area bounded by* f *and x-axis from -1 to 1. Which statements are true?*

- *I:* f *continuous in [-1,1]*
- *II:* f *not bounded in [-1,1]*
- *III:* f *non-zero and finite*

Options:

- *A: II only*
- *B: III only*
- *C: II and III only*
- *D: I, II, III*

GATEPYQ 17 *Evaluate*

$$\lim_{x\to\infty} x^{1/x}$$

- *A:* ∞
- *B: 0*
- *C: 1*
- *D: Not defined*

GATEPYQ 18 *If* $f(x) = x\sin x$ *satisfies*

$$f''(x) + f(x) + t\cos x = 0$$

then find t.

GATEPYQ 19 *Find the least value of* $f(x) = 2x^2 - 8x - 3$ *in* $[0, 5]$.

- *A: -15*
- *B: 7*
- *C: -11*
- *D: -3*

GATEPYQ 20 *Which of the following functions is continuous at* $x = 3$*?*

- *A:* $f(x) = \begin{cases} 2, & x = 3 \\ x - 1, & x > 3 \\ \frac{x+3}{3}, & x < 3 \end{cases}$
- *B:* $f(x) = \begin{cases} 1, & x = 3 \\ x^2 - 9, & x \neq 3 \end{cases}$
- *C:* $f(x) = \frac{x^2-9}{x-3}$
- *D:* $f(x) = \sin(x - 3)$

GATEPYQ 21 *Find* $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$.

GATEPYQ 22 *Evaluate*

$$\lim_{x\to 0} \frac{\sin 5x}{\sin 3x}.$$

GATEPYQ 23 *Evaluate*

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}.$$

7.2 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
18	Chapter 7 — GATE PYQs Solutions 1–23	https://youtu.be/06yBEx4ITrg	

Chapter 8

Solutions

Problems Covered	YouTube Link	QR Code
Solutions to Problems 1–5	https://youtu.be/OwEf3BBTK9A	
Solutions to Problems 6–19	https://youtu.be/3mbWTr3XspE	
Solutions to Problems 20–29	https://youtu.be/XxFsEutFMv0	

Solutions to Problems 30–39	https://youtu.be/iqr-e1rSjUQ	
Solutions to Problems 40–47	https://youtu.be/xc4m8MS5J88	
Solutions to Problems 48–52	https://youtu.be/q5n3QCe4ICQ	
GATE PYQs Solutions 1–23	https://youtu.be/06yBEx4ITrg	

Bibliography

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