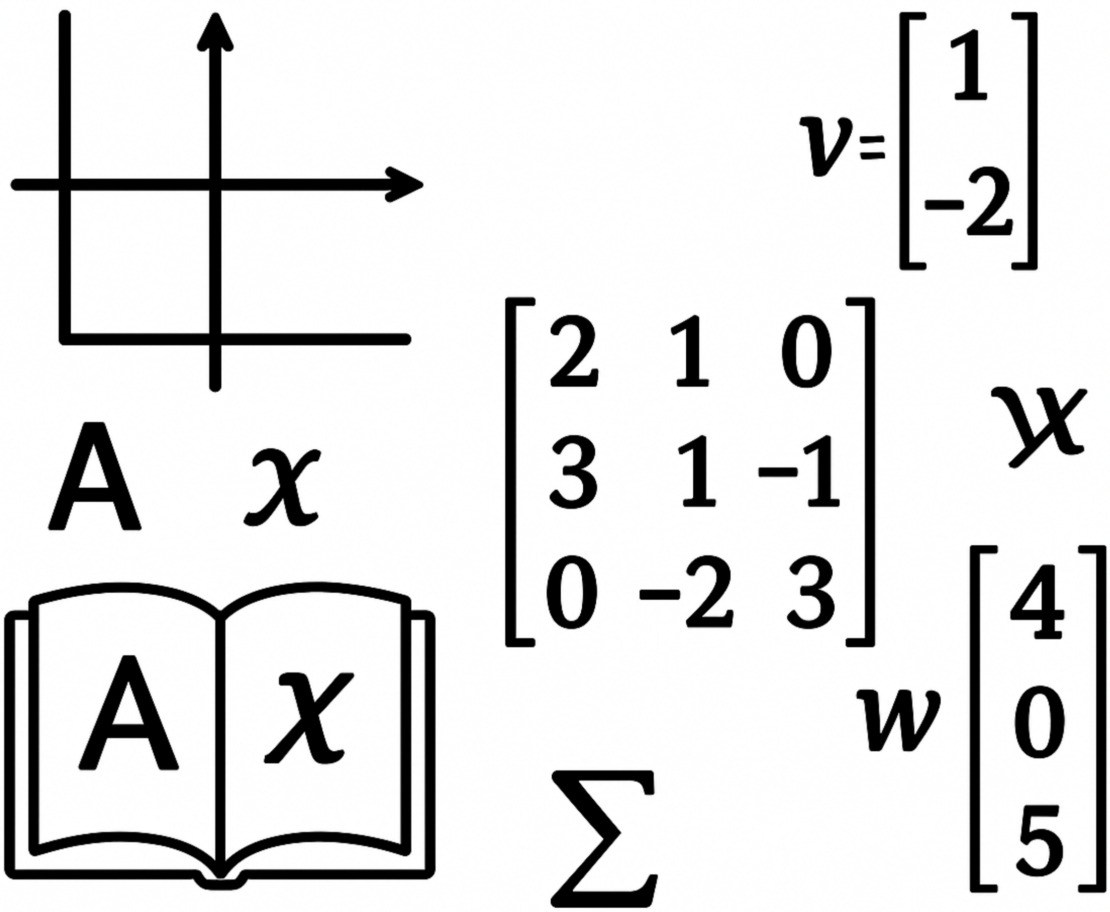
GATE – Data Science and Artificial Intelligence (DA)

Vectors and Linear Algebra



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About the Book

Artificial Intelligence and Machine Learning (AI/ML) are transforming industries across the globe — from healthcare and finance to transportation and education. From medical diagnosis systems and fraud detection to personalized recommendations and autonomous vehicles, AI/ML is shaping the way we live, work, and interact with technology.

To support this rapidly growing field, the GATE Data Science and Artificial Intelligence (DA) exam was introduced as a national-level gateway to higher studies, research, and employment opportunities in top institutions and organizations. The exam tests a candidate's proficiency in mathematics, programming, data handling, machine learning, and AI fundamentals.

This book is a compact and comprehensive guide for GATE DA aspirants. It is designed to help learners build a strong conceptual foundation while developing the problem-solving skills required for the exam. Many solved examples are included to illustrate key concepts, and each chapter features carefully crafted problems for practice.

Solutions to selected problems and topic-wise lectures will be discussed in detail on my YouTube channel (@GATEXAiml). All the concepts covered in the book will also be taught step-by-step through video tutorials, making this a complete learning resource for GATE DA preparation.

This book is designed for aspirants of the GATE DA exam focusing on **Vectors and Linear Algebra**. It systematically covers theory, solved examples, and practice problems aligned with the official syllabus.

Dedicated to all my Gurus and Students.
"Knowledge grows only when shared — and it must remain free, for that is how it thrives."

Linear Algebra (Vectors and Linear Algebra) - Syllabus

Vector space, subspaces, linear dependence and independence of vectors, matrices, quadratic forms, projections onto subspace, singular value decomposition.

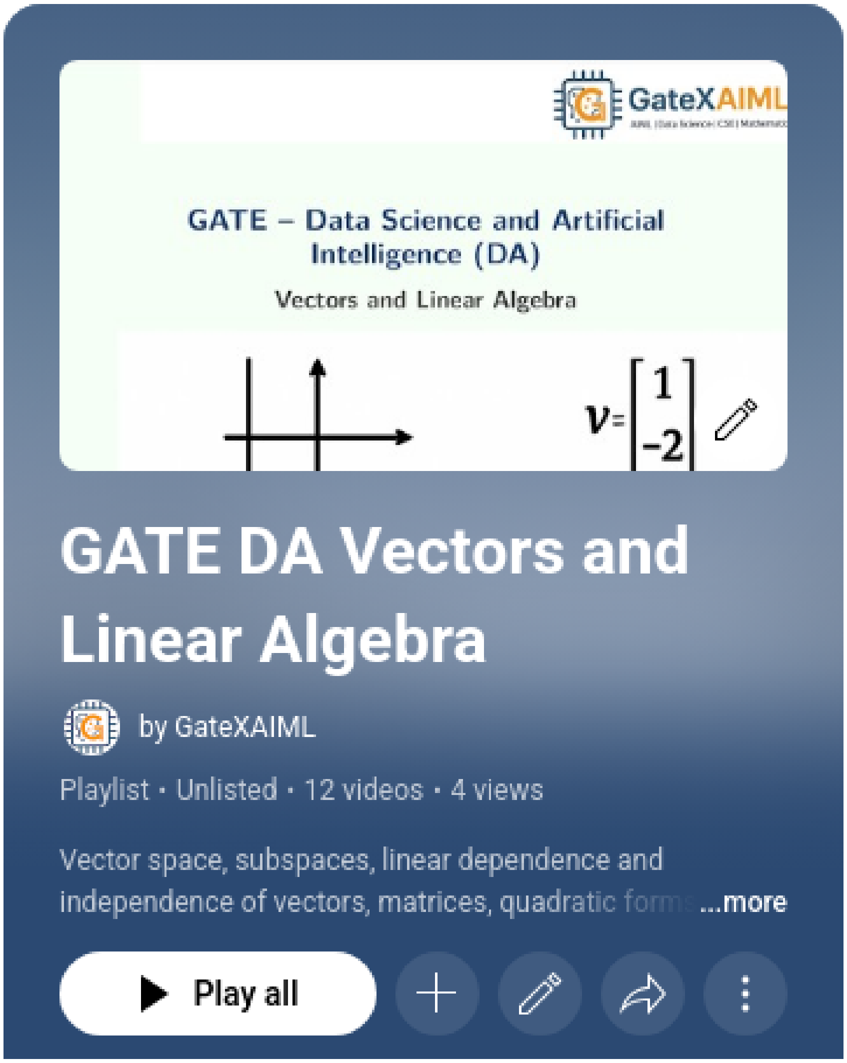
STOP!

Attention!

Some examples solved in video lectures are different from those given in this book.

The procedure to solve problems and examples is well explained in the video lectures, and it is highly recommended to go through the video lectures for complete understanding.

Official Video Playlist



Watch on YouTube

Chapter 1

Vector Spaces

1.1 Vectors Concept

What is a Vector?

Concept

A *vector* is a mathematical object that has both magnitude and direction. In coordinate form, a vector in $\mathbb{R}^n$ is an ordered tuple

$$\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n.$$

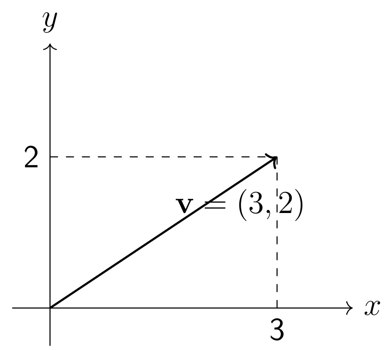
Vectors are usually written in bold (e.g. $\mathbf{v}$) or with an arrow overhead ($\vec{v}$). A *zero vector* has all components zero and is denoted $\mathbf{0}$.

Geometric picture and basic diagrams

Concept

A vector in 2D can be drawn as a directed arrow from the origin to the point (x, y). Vectors can be translated (moved) without changing their meaning as long as direction and magnitude remain same.

Example 1: 2D vector diagram



Length or Magnitude of a Vector

Concept

The **length (or magnitude or norm)** of a vector $\mathbf{v}$ represents how long the vector is — that is, its distance from the origin to the point it represents in space.
If $\mathbf{v} = (v_1, v_2, v_3, \ldots, v_n)$ in $\mathbb{R}^n$, then the magnitude of $\mathbf{v}$ is given by

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2}.$$

This follows from the **Pythagorean theorem** generalized to n dimensions.
Properties:

- $\|\mathbf{v}\| \geq 0$ for all vectors.
- $\|\mathbf{v}\| = 0 \iff \mathbf{v} = \mathbf{0}$.
- $\|k\mathbf{v}\| = |k|\,\|\mathbf{v}\|$ for any scalar k.
- The distance between two points $\mathbf{A}$ and $\mathbf{B}$ is $\|\mathbf{A} - \mathbf{B}\|$.

Example 2: Magnitude in 2D and 3D

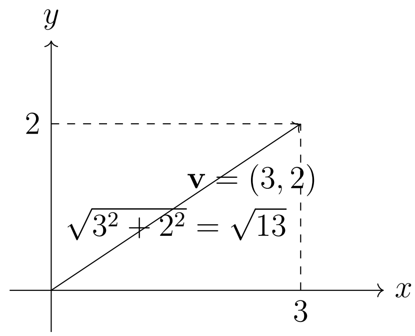
1. For $\mathbf{v} = (3, 4)$,
$$\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = 5.$$
2. For $\mathbf{v} = (1, 2, 2)$,
$$\|\mathbf{v}\| = \sqrt{1^2 + 2^2 + 2^2} = 3.$$

Example 3: Distance between two points

Let $A(1, 2, 3)$ and $B(4, 6, 9)$. Then vector $\overrightarrow{AB} = B - A = (3, 4, 6)$, and

$$\text{Distance} = \left\|\overrightarrow{AB}\right\| = \sqrt{3^2 + 4^2 + 6^2} = \sqrt{61}.$$

Example 4: Geometric visualization



Example 5: Simple vector

Let $\mathbf{a} = (3, -1, 2)$. Then $\mathbf{a}$ is a vector in $\mathbb{R}^3$ with magnitude

$$\|\mathbf{a}\| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}.$$

Vector arithmetic

Concept

Given $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ and scalar $\alpha \in \mathbb{R}$:

- Addition: $\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ldots, u_n + v_n)$.
- Scalar multiplication: $\alpha\mathbf{v} = (\alpha v_1, \ldots, \alpha v_n)$.
- Subtraction: $\mathbf{u} - \mathbf{v} = (u_1 - v_1, \ldots, u_n - v_n)$.

Vector operations satisfy commutativity, associativity, distributivity, existence of zero vector and additive inverses.

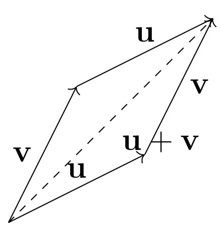
Example 6: Addition and scalar multiple

Let $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (3, -1)$. Then

$$\mathbf{u} + \mathbf{v} = (4, 1), \qquad 2\mathbf{u} = (2, 4).$$

Geometric interpretation: addition corresponds to placing vectors head-to-tail (parallelogram rule).

Example 7: Parallelogram



Dot product (Inner product) and properties

Concept

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the dot product (inner product) is

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i.$$

Properties:

1. Symmetry: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
2. Linearity: $(a\mathbf{u} + b\mathbf{w}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v}) + b(\mathbf{w} \cdot \mathbf{v})$.
3. Positive-definiteness: $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 \geq 0$ and equals zero only for $\mathbf{v} = \mathbf{0}$.
4. Relation to angle: $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$, where θ is the angle between them.

Example 8: Angle between vectors

Let $\mathbf{a} = (1, 1, 0)$ and $\mathbf{b} = (1, 0, 1)$. Then

$$\mathbf{a} \cdot \mathbf{b} = 1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 = 1,$$

$$\|\mathbf{a}\| = \sqrt{2}, \quad \|\mathbf{b}\| = \sqrt{2} \quad \Rightarrow \quad \cos\theta = \frac{1}{2}.$$

So $\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$.

Norms and distances

Concept

The (Euclidean) norm of $\mathbf{v} \in \mathbb{R}^n$ is $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. Distance between points $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ is $\|\mathbf{p} - \mathbf{q}\|$.
Triangle inequality:

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Cauchy–Schwarz inequality:

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|.$$

Example 9: Triangle inequality check

Take $\mathbf{u} = (1, 0)$ and $\mathbf{v} = (0, 1)$. Then $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$ and $\|\mathbf{u} + \mathbf{v}\| = \sqrt{2} \leq 2$ holds.

Cross product (3D) and properties

Concept

For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, the cross product $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to both, given by the determinant formula:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Properties:

- $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$ (antisymmetric).
- $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta$ (area of parallelogram).
- $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$.

Cross product is defined only in 3D (or via wedge product in higher dimensional algebra).

Example 10: Cross product value

Let $\mathbf{a} = (1, 0, 0)$ and $\mathbf{b} = (0, 1, 0)$. Then $\mathbf{a} \times \mathbf{b} = (0, 0, 1)$.

Example 11: Area of triangle using cross product

Triangle with vertices $A(1, 0, 0)$, $B(2, 1, 0)$, $C(1, 1, 1)$. Area $= \frac{1}{2} \left\| (\overrightarrow{AB} \times \overrightarrow{AC}) \right\|$. Compute $\overrightarrow{AB} = (1, 1, 0)$, $\overrightarrow{AC} = (0, 1, 1)$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = (1 \cdot 1 - 0 \cdot 1)\mathbf{i} - (1 \cdot 1 - 0 \cdot 0)\mathbf{j} + (1 \cdot 1 - 1 \cdot 0)\mathbf{k} = (1, -1, 1).$$

Norm $= \sqrt{3}$, area $= \frac{1}{2}\sqrt{3}$.

1.2 Vector Space

Concept

Elimination simplifies the system $A\mathbf{x} = \mathbf{b}$, one entry at a time. It also simplifies the underlying theory — particularly the fundamental questions of **existence** and **uniqueness** of solutions:

- Does the system have one unique solution?

- Does it have no solution?
- Or does it have infinitely many solutions?

After elimination, these questions become easier to answer. However, elimination gives only one kind of understanding of $A\mathbf{x} = \mathbf{b}$. Linear algebra goes deeper — it explores the very structure of the space that these vectors and equations belong to.

Concept of a Vector Space

Concept

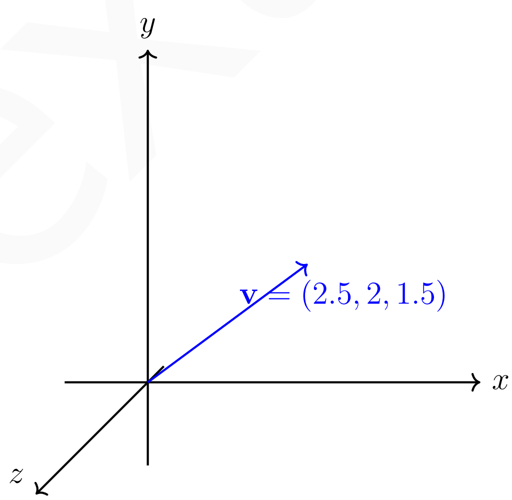
To develop a deeper understanding, we begin with the most fundamental spaces, denoted by:

$$\mathbb{R}^1,\ \mathbb{R}^2,\ \mathbb{R}^3, \ldots, \mathbb{R}^n$$

Each $\mathbb{R}^n$ consists of all column vectors with n real components.

- $\mathbb{R}^1$ — all real numbers (a line)
- $\mathbb{R}^2$ — the usual xy-plane
- $\mathbb{R}^3$ — three-dimensional space

The two components of a vector in $\mathbb{R}^2$ represent the x and y coordinates of a point in the plane. Similarly, the three components of a vector in $\mathbb{R}^3$ represent a point in three-dimensional space.



Generalization to Higher Dimensions

Concept

The beauty of linear algebra lies in how naturally it extends to higher dimensions. A vector in $\mathbb{R}^7$, for example, is represented simply by its seven components:

$$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$$

Even though we cannot visualize 7-dimensional geometry, all algebraic operations remain the same:

- Vector addition: $\mathbf{x} + \mathbf{y}$
- Scalar multiplication: $c\mathbf{x}$

Definition of a Real Vector Space

Concept

A **real vector space** is a set of elements (vectors) together with two operations:

- Vector addition
- Scalar multiplication by real numbers

These operations must:

1. Produce vectors that remain inside the space.
2. Satisfy the eight basic vector space axioms.

Eight Basic Axioms of Vector Spaces

Concept

1. **Closure under addition:** For all $\mathbf{u}, \mathbf{v} \in V$, the sum $\mathbf{u} + \mathbf{v} \in V$.
2. **Commutative property of addition:** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. **Associative property of addition:** $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. **Existence of additive identity:** There exists a zero vector $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
5. **Existence of additive inverse:** For every $\mathbf{v} \in V$, there exists $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
6. **Closure under scalar multiplication:** For all $c \in \mathbb{R}$ and $\mathbf{v} \in V$, $c\mathbf{v} \in V$.
7. **Distributive laws:**

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v} \qquad \textbf{(scalar over vector addition)}$$
$$(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v} \qquad \textbf{(scalar addition over vector)}$$

8. **Compatibility with scalar multiplication:** $c(d\mathbf{v}) = (cd)\mathbf{v}$ and $1\mathbf{v} = \mathbf{v}$.

Example 12: Verifying the Axioms in $\mathbb{R}^2$

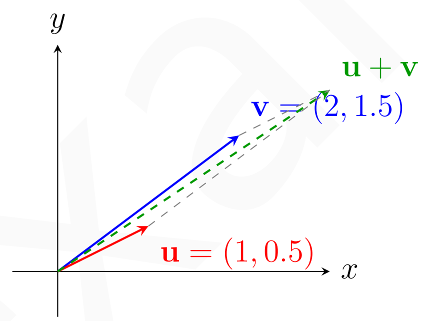
Let $V = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. Define addition and scalar multiplication as:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2), \quad c(x, y) = (cx, cy)$$

Let us verify some axioms:

- **Closure under addition:** If $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, then $(x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2$.
- **Existence of additive identity:** $(0, 0)$ is the zero vector since $(x, y) + (0, 0) = (x, y)$.
- **Existence of additive inverse:** The inverse of (x, y) is $(-x, -y)$ because $(x, y) + (-x, -y) = (0, 0)$.
- **Distributive property:** $c((x_1, y_1) + (x_2, y_2)) = c(x_1 + x_2, y_1 + y_2) = (c(x_1 + x_2), c(y_1 + y_2)) = (cx_1 + cx_2, cy_1 + cy_2) = c(x_1, y_1) + c(x_2, y_2)$.

Thus, $\mathbb{R}^2$ satisfies all eight vector space axioms.



Geometric interpretation: Vector addition in $\mathbb{R}^2$ corresponds to the parallelogram law.

1.3 Subspaces

Concept

A **subspace** of a vector space V is a nonempty subset $W \subseteq V$ that itself satisfies the conditions of a vector space. That is, W is closed under linear combinations:

1. If $\mathbf{x}, \mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.
2. If $c \in \mathbb{R}$ and $\mathbf{x} \in W$, then $c\mathbf{x} \in W$.

Thus, a subspace is a subset that is **closed** under vector addition and scalar multiplication. Because these operations already satisfy the eight vector space axioms in V, they automatically hold in any subspace W.
Note: Every subspace must contain the zero vector, since if $\mathbf{x} \in W$, then $0\mathbf{x} = \mathbf{0} \in W$.

Concept

The possible subspaces of $\mathbb{R}^3$ include:

- The **zero subspace** $\{\mathbf{0}\}$
- Any **line through the origin**
- Any **plane through the origin**
- The entire space $\mathbb{R}^3$

The smallest subspace is $\{\mathbf{0}\}$, containing only the origin. The largest subspace is the whole space $\mathbb{R}^3$ itself.

Example 13: Subset vs. Subspace

Case 1: Consider all vectors in $\mathbb{R}^2$ with nonnegative components:

$$W = \{(x, y) \mid x \geq 0,\, y \geq 0\}$$

This set is the **first quadrant**. It contains the zero vector and is closed under addition, but not under scalar multiplication. If we multiply $(1, 1)$ by -1, we get $(-1, -1)$, which lies outside the first quadrant. Hence, W is **not** a subspace.
Case 2: If we include both the first and third quadrants, multiplication by negative scalars is allowed, but addition can move vectors out of these quadrants. Thus, even this enlarged set is not a subspace.
The smallest subspace containing the first quadrant is the entire $\mathbb{R}^2$.

Example 14: Subspaces of Matrix Spaces

Let V be the vector space of all 3×3 matrices. Then the following are subspaces of V:

- The set of all **lower triangular matrices**
- The set of all **symmetric matrices**

Both sets are closed under matrix addition and scalar multiplication, and contain the zero matrix.

Concept

For a matrix $A \in \mathbb{R}^{m\times n}$, the **column space** of A, denoted $C(A)$, is the set of all linear combinations of the columns of A:

$$C(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$$

It is a subspace of $\mathbb{R}^m$.
The system $A\mathbf{x} = \mathbf{b}$ is **solvable** if and only if $\mathbf{b} \in C(A)$. That is, $\mathbf{b}$ can be expressed as a linear combination of the columns of A.

Example 15: Column Space in $\mathbb{R}^3$

Consider

$$A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Then $A\mathbf{x} = \mathbf{b}$ means

$$u \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + v \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

All attainable right-hand sides $\mathbf{b}$ lie in the **plane spanned** by the two column vectors of A. This plane is the column space $C(A) \subseteq \mathbb{R}^3$.

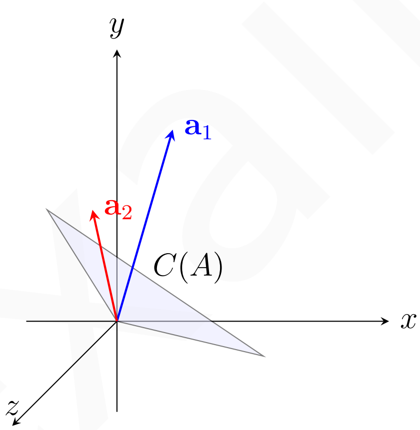


Figure: Column space $C(A)$ is a plane through the origin in $\mathbb{R}^3$.

Concept

The **nullspace** of a matrix A, denoted $N(A)$, is the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{0}.$$

It is a subspace of $\mathbb{R}^n$.
Properties:

1. If $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x}' = \mathbf{0}$, then $A(\mathbf{x} + \mathbf{x}') = \mathbf{0}$.
2. If $A\mathbf{x} = \mathbf{0}$, then $A(c\mathbf{x}) = \mathbf{0}$ for any scalar c.

Note: Only homogeneous systems ($\mathbf{b} = 0$) form subspaces.

Example 16: Nullspace Calculation

For

$$A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix}, \quad A\mathbf{x} = \mathbf{0} \Rightarrow \begin{cases} u = 0 \\ 5u + 4v = 0 \\ 2u + 4v = 0 \end{cases} \Rightarrow u = v = 0.$$

Hence, $N(A) = \{(0,0)\}$. The columns are linearly independent.
Now let

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix}.$$

Then

$$B \begin{bmatrix} c \\ c \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, $N(B) = \{c(1,1,-1) \mid c \in \mathbb{R}\}$, which is a **line through the origin** — a one-dimensional subspace.

Concept

- The **column space** $C(A) \subseteq \mathbb{R}^m$ contains all attainable right-hand sides $\mathbf{b}$ for $A\mathbf{x} = \mathbf{b}$.
- The **nullspace** $N(A) \subseteq \mathbb{R}^n$ contains all solutions $\mathbf{x}$ to $A\mathbf{x} = 0$.
- Both are subspaces; they play a fundamental role in understanding linear systems.

1.4 Linear Independence, Basis, and Dimension

Concept

Linear Independence: A set of vectors $\{v_1, v_2, \ldots, v_k\}$ is *linearly independent* if the equation

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

has only the trivial solution $c_1 = c_2 = \cdots = c_k = 0$.
If there exists a nontrivial solution (some $c_i \neq 0$), the vectors are *linearly dependent*.

Example 17:

Let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$.
Check $c_1v_1 + c_2v_2 + c_3v_3 = 0$:

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving gives $c_1 = c_2 = c_3 = 0$.
Hence, $\{v_1, v_2, v_3\}$ are **linearly independent**.

Example 18:

Let $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ in $\mathbb{R}^2$.
Check $c_1v_1 + c_2v_2 = 0$:

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A nontrivial solution exists: $c_1 = 2, c_2 = -1$.
Hence, v_1 and v_2 are **linearly dependent**.

Concept

Spanning a Subspace: A set of vectors $\{w_1, \ldots, w_\ell\}$ spans a subspace V if every vector $v \in V$ can be written as

$$v = c_1w_1 + c_2w_2 + \cdots + c_\ell w_\ell$$

for some scalars $c_1, \ldots, c_\ell$.

Example 19:

In $\mathbb{R}^3$, vectors $w_1 = (1, 0, 0)$, $w_2 = (0, 1, 0)$, $w_3 = (1, 1, 0)$ span the x-y plane.
Even w_1 and w_2 alone span the plane. w_1 and w_3 only span a line.

Example 20:

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$. The columns of A span a subspace of $\mathbb{R}^3$. Any vector v in this subspace can be written as $v = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Concept

Basis of a Vector Space: A basis of a vector space V is a set of vectors that

1. are linearly independent, and
2. span V.

Every vector in V can then be expressed uniquely as a linear combination of basis vectors.

Example 21:

In $\mathbb{R}^2$, vectors $v_1 = (1, 2)$, $v_2 = (2, 3)$ are linearly independent and span $\mathbb{R}^2$.
Hence, $\{v_1, v_2\}$ is a basis of $\mathbb{R}^2$.

Example 22:

For the matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, pivot columns are 1, 2, 3. These columns form a basis of the column space of A.

Concept

Dimension of a Vector Space: The number of vectors in any basis of a vector space V is called the *dimension* of V, denoted $\dim(V)$.
All bases of V have the same number of vectors.

Example 23:

- R^3 has dimension 3. Standard basis: $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. - x-y plane in $\mathbb{R}^3$ has dimension 2.

Example 24:

Matrix $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has pivot columns 1 and 3. Dimension of the column space = 2.

Concept

Rank and Linear Independence:

- The **rank** of a matrix A = number of independent rows or columns.
- Rank = dimension of column space = dimension of row space.
- If $n > m$ in an $m \times n$ matrix, the n columns are dependent.

Example 25:

Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$. - $m = 2, n = 3$, so $n > m$. - Rank = 1 (only first column is independent). - All 3 columns are linearly dependent.

Example 26:

Let $U = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. - Pivot columns: 1 and 2. - Basis for column space: columns 1 and 2. - Dimension = rank = 2.

Concept

Key Theorems:

1. Any set of more than n vectors in $\mathbb{R}^n$ is linearly dependent.
2. A linearly independent set can be extended to a basis.

3. A spanning set can be reduced to a basis.
4. All bases of a vector space have the same number of vectors.

1.5 Applications of Vector Spaces

Solving Systems of Linear Equations

Vector spaces provide a framework to understand solutions of linear systems $Ax = b$.
- The column space $C(A)$ determines whether b lies in the span of the columns. - The nullspace $N(A)$ gives all solutions of the homogeneous system $Ax = 0$. - The rank of A tells the number of independent equations. - A basis of $C(A)$ allows expressing any solution as a linear combination of independent columns.

Computer Graphics

Vector spaces represent points, directions, and transformations in 2D/3D graphics.
- Any point or vector can be expressed as a combination of basis vectors. - Linear transformations like rotations, scaling, and projections are matrix operations on vector spaces.

Signal Processing

Signals are represented as vectors in function spaces or discrete vector spaces.
- Orthogonal bases, such as Fourier or wavelet bases, allow decomposition into independent components. - The dimension of the vector space corresponds to the number of independent signal components.

Computer Science and Data Analysis

Feature vectors in machine learning exist in high-dimensional vector spaces.
- Linear independence ensures that features are non-redundant. - Basis vectors can be used for dimensionality reduction, e.g., in Principal Component Analysis (PCA). - Rank of the data matrix reveals the intrinsic dimensionality of the dataset.

Physics and Engineering

Vector spaces describe physical quantities such as forces, velocities, and fields.
- Basis vectors often correspond to orthogonal coordinate axes. - Linear independence ensures that combinations of forces or fields can be uniquely represented. - The dimension of the space corresponds to the number of independent physical directions.

Control Systems

State-space models use vectors for system states and inputs.
- Linear independence of state vectors is crucial for controllability and observability. - Dimension of the state-space corresponds to the number of independent states of the system.

1.6 Conceptual GATE PYQs

GATEPYQ 1 *Let A be any $n \times m$ matrix, where $m > n$.*
Which of the following statements is/are TRUE about the system of linear equations $Ax = 0$?

(A) There exist at least $m - n$ linearly independent solutions to this system
(B) There exist $m - n$ linearly independent vectors such that every solution is a linear combination of these vectors
(C) There exists a non-zero solution in which at least $m - n$ variables are 0
(D) There exists a solution in which at least n variables are non-zero

GATEPYQ 2 *Let $c_1, \ldots, c_n$ be scalars, not all zero, such that $\sum_{i=1}^{n} c_i a_i = 0$ where a_i are column vectors in $\mathbb{R}^n$. Consider the set of linear equations $Ax = b$ where $A = [a_1 \ldots a_n]$ and $b = \sum_{i=1}^{n} a_i$. The set of equations has:*

(A) a unique solution at $x = J_n$, where J_n denotes an n-dimensional vector of all 1's
(B) no solution
(C) infinitely many solutions
(D) finitely many solutions

GATEPYQ 3 *Consider the systems, each consisting of m linear equations in n variables:*
I. If $m \ll n$, then all such systems have a solution
II. If $m \gg n$, then none of these systems has a solution
III. If $m = n$, then there exists a system which has a solution
Which one of the following is CORRECT?

(A) I, II and III are true
(B) Only II and III are true
(C) Only III is true
(D) None of them is true

GATEPYQ 4 *If M is a square matrix with zero determinant, which of the following assertions are correct?*
S1: Each row of M can be represented as a linear combination of the other rows
S2: Each column of M can be represented as a linear combination of the other columns
S3: $MX = 0$ has a nontrivial solution
S4: M has an inverse
(A) S3 and S2
(B) S1 and S4
(C) S1 and S3
(D) S1, S2 and S3

GATEPYQ 5 *F is an $n \times n$ real matrix, b is an $n \times 1$ real vector. Suppose there are two vectors $u \neq v$ such that $Fu = b$ and $Fv = b$. Which statement is FALSE?*

(A) Determinant of F is zero
(B) There are an infinite number of solutions to $Fx = b$
(C) There is an $x \neq 0$ such that $Fx = 0$
(D) F must have two identical rows

GATEPYQ 6 *Let $AX = b$ be a system with A an $m \times n$ matrix, b an $m \times 1$ vector, and X an $n \times 1$ vector of unknowns. Which is FALSE?*

(A) The system has a solution iff A and augmented matrix $[A|b]$ have the same rank
(B) If $m < n$ and $b = 0$, the system has infinitely many solutions
(C) If $m = n$ and $b \neq 0$, the system has a unique solution
(D) The system will have only trivial solution when $m = n$, $b = 0$ and rank$(A) = n$

1.7 Problems

Problem 1 *(MCQ)*

Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 1 + 6\sqrt{\frac{3}{5}} \\ 2 - 3\sqrt{\frac{3}{5}} \\ 2 \end{bmatrix}$. *The angle θ between $\mathbf{u}$ and $\mathbf{v}$ is:*

A. $30°$
B. $45°$
C. $60°$
D. $90°$

Problem 2 *(MCQ) The cross product $\mathbf{u} \times \mathbf{v}$ of* $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ *is:*

A. $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

C. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Problem 3 *(NAT) If* $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, *and* $\mathbf{w} = \mathbf{u} + k\mathbf{v}$ *is perpendicular to* $\mathbf{u}$, *the value of* k *is:*

Problem 4 *(MCQ) If* $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, *then the cross product* $\mathbf{u} \times \mathbf{v}$*:*

A. Is zero vector
B. Is parallel to $\mathbf{u}$
C. Is parallel to $\mathbf{v}$
D. Is perpendicular to both $\mathbf{u}$ *and* $\mathbf{v}$

Problem 5 *(MCQ) If* $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 1 \\ \\ 1 \end{bmatrix}$, *then the magnitude of* $\mathbf{u} \times \mathbf{v}$ *is:*

A. 0
B. $\sqrt{6}$
C. $\sqrt{3}$
D. 1

Problem 6 *(MCQ) Which of the following statements about vector spaces is true?*
A. The zero vector cannot belong to a vector space
B. The sum of two vectors in a vector space is always in the vector space
C. Multiplying a vector by a scalar always gives a vector outside the space
D. Vector spaces can have only finite number of vectors

Problem 7 *(MCQ) The dimension of the subspace of* $\mathbb{R}^4$ *defined by* $x_1 + x_2 + x_3 + x_4 = 0$ *is:*
A. 1
B. 2
C. 3
D. 4

Problem 8 *(MSQ) Let* V *be the set of all polynomials of degree at most 2. Which of the following are bases of* V*?*
A. $\{1, x, x^2\}$
B. $\{1, x, x^2 + 1\}$
C. $\{1, x^2, x^3\}$
D. $\{x, x^2, x^2 + 1\}$

Problem 9 *(MCQ) Which of the following is always a basis of* $\mathbb{R}^2$*?*
A. $\{(1, 0), (0, 1)\}$
B. $\{(1, 1), (2, 2)\}$
C. $\{(1, 2), (3, 4)\}$
D. $\{(0, 0), (1, 0)\}$

Problem 10 *(NAT) The number of vectors in any basis of* $\mathbb{R}^4$ *is:*

Problem 11 *(NAT) If a matrix* A *is* 4×5 *with rank* 3*, the dimension of its column space is:*

Problem 12 *(MSQ) Which of the following sets are linearly independent?*

A. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

C. $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}$

D. $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}$

Problem 13 *(MCQ) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^3$ are linearly dependent, which of the following is always true?*
A. *One vector is zero*
B. *One vector can be written as combination of others*
C. *All vectors are zero*
D. *Cross product $\mathbf{v}_1 \times \mathbf{v}_2$ is zero*

Problem 14 *(MSQ) Which of the following sets of vectors in $\mathbb{R}^3$ are linearly independent?*
A. $\{(1,0,0),(0,1,0),(0,0,1)\}$
B. $\{(1,2,3),(2,4,6),(3,6,9)\}$
C. $\{(1,0,1),(0,1,1),(1,1,0)\}$
D. $\{(1,1,1),(2,2,2),(3,3,3)\}$

Problem 15 *(MSQ) Which of the following sets are linearly dependent?*

A. $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$

B. $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}$

C. $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$

D. $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$

Problem 16 *(MCQ) If v_1, v_2, v_3, v_4 are vectors in $\mathbb{R}^3$, then:*
A. They are always independent
B. They are always dependent
C. They span $\mathbb{R}^3$
D. None of the above

Problem 17 *(NAT) If the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent in $\mathbb{R}^3$, what is the maximum number of linearly independent vectors that can exist in $\mathbb{R}^3$?*

Problem 18 *(MSQ) Which of the following sets span $\mathbb{R}^3$?*
A. $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$
B. $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$
C. $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix}$
D. $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$

Problem 19 *(MSQ) Which of the following sets of vectors form a basis of $\mathbb{R}^3$?*
A. $\{(1,0,0),(0,1,0),(0,0,1)\}$
B. $\{(1,2,3),(4,5,6),(7,8,9)\}$
C. $\{(1,0,1),(0,1,1),(1,1,0)\}$

D. $\{(1,1,1),(2,2,2),(3,3,3)\}$

Problem 20 *(NAT) The determinant of the matrix formed by placing the vectors* $\mathbf{v}_1 = (1,0,1)$, $\mathbf{v}_2 = (0,1,1)$, *and* $\mathbf{v}_3 = (1,1,0)$ *as columns is:*

Problem 21 *(MCQ) Which of the following vectors is in the nullspace of* $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$*?*

A. $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

D. $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

Problem 22 *(NAT) The rank of the matrix* $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & 1 \end{bmatrix}$ *is:*

Problem 23 *(NAT) If* A *is* 3×4 *and rank*$(A) = 2$*, the dimension of its nullspace is:*

Problem 24 *(MSQ) Which of the following sets span the column space of* $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$*?*

A. Columns of A
B. Rows of A
C. $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$
D. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Problem 25 *(MSQ) Which of the following are subspaces of* $\mathbb{R}^3$*?*
A. $\{(x, y, z)|x + y + z = 0\}$
B. $\{(x, y, z)|x > 0, y > 0, z > 0\}$
C. $\{(x, y, z)|x = y = z\}$
D. $\{(0, 0, 0)\}$

Problem 26 *(NAT) The dimension of the space of all* 2×2 *symmetric matrices is:*

Problem 27 *(MSQ) The cross product* $\mathbf{u} \times \mathbf{v}$ *is zero if:*
A. $\mathbf{u} = \mathbf{0}$
B. $\mathbf{v} = \mathbf{0}$
C. $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel*
D. $\mathbf{u}$ *and* $\mathbf{v}$ *are perpendicular*

1.8 Try it Yourself

Exercise 1 *(MCQ) If* $\mathbf{a} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ *and* $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$*, the angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *is:*

A. 30°

B. 45°
C. 60°
D. 90°

Exercise 2 *(MCQ) The length of* $\mathbf{v} = \begin{bmatrix} -2 \\ 5 \\ 6 \end{bmatrix}$ *is:*
A. 7
B. 8
C. $\sqrt{65}$
D. 10

Exercise 3 *(MCQ) If* $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$*, compute* $\mathbf{u} \cdot \mathbf{v}$*:*
A. 2
B. 4
C. 5
D. 6

Exercise 4 *(MCQ) The cross product* $\mathbf{i} \times \mathbf{k}$ *is:*
A. $\mathbf{j}$
B. $-\mathbf{j}$
C. $\mathbf{i}$
D. $\mathbf{k}$

Exercise 5 *(NAT) If* $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$*, find scalar projection of* $\mathbf{u}$ *on* $\mathbf{v}$*.*

Exercise 6 *(MCQ) Which of the following is true for any vector space* V*?*
A. $0 \notin V$
B. $v + w \in V$ for any $v, w \in V$
C. $kv \notin V$ for scalar k
D. Vector space has only finite elements

Exercise 7 *(MCQ) Dimension of subspace of* $\mathbb{R}^3$ *defined by* $x + y + z = 0$ *is:*
A. 1
B. 2
C. 3
D. 4

Exercise 8 *(MSQ) Which of the following sets are bases for polynomials of degree* ≤ 2*?*
A. $\{1, x, x^2\}$
B. $\{1, x, x^2 + 1\}$
C. $\{1, x^2, x^3\}$
D. $\{x, x^2, x^2 + 1\}$

Exercise 9 *(NAT) Maximum number of linearly independent vectors in* $\mathbb{R}^4$ *is:*

Exercise 10 *(MCQ) The set* $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ *is:*
A. Linearly independent
B. Linearly dependent
C. Spans $\mathbb{R}^3$
D. None of the above

Exercise 11 *(MSQ) Which of the following sets are linearly dependent?*
A. $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$
B. $\{(1, 2, 3), (2, 4, 6), (3, 6, 9)\}$
C. $\{(1, 0, 1), (0, 1, 1), (1, 1, 0)\}$
D. $\{(2, 3, 5), (4, 6, 10)\}$

Exercise 12 *(MCQ) If* v_1, v_2, v_3 *in* $\mathbb{R}^3$ *are linearly dependent, which is always true?*
A. $v_1 = 0$
B. One is combination of others
C. All zero
D. Cross product $v_1 \times v_2 = 0$

Exercise 13 *(NAT) Find* k *such that* $\mathbf{w} = \mathbf{u} + k\mathbf{v}$ *is perpendicular to* $\mathbf{u}$*, where* $\mathbf{u} = [1, 1, 1]^T$*,* $\mathbf{v} = [2, 0, 1]^T$*.*

Exercise 14 *(MCQ) Four vectors in* $\mathbb{R}^3$ *are:*
A. Always independent
B. Always dependent
C. Span $\mathbb{R}^3$
D. None of the above

Exercise 15 *(NAT) Dimension of nullspace of* 3×5 *matrix with rank 2:*

Exercise 16 *(MSQ) Which sets span* $\mathbb{R}^3$*?*
A. Standard basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$
B. $\{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$
C. $\{(1, 2, 3), (2, 4, 6), (3, 6, 9)\}$
D. $\{(1, 0, 0), (0, 1, 0)\}$

Exercise 17 *(MCQ) Rank of* $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & 1 \end{bmatrix}$ *is:*
A. 1

B. 2
C. 3
D. 0

Exercise 18 *(NAT) Dimension of column space of* 4×5 *matrix with rank 3:*

Exercise 19 *(MCQ) Vector* $\mathbf{v} = [1, 2, 3]^T$ *belongs to nullspace of* $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$*?*

A. Yes
B. No
C. Always
D. Zero vector only

Exercise 20 *(MCQ) Magnitude of* $\mathbf{u} \times \mathbf{v}$ *for* $\mathbf{u} = [1, 0, 0]$*,* $\mathbf{v} = [1, 1, 0]$ *is:*
A. 0
B. 1
C. $\sqrt{2}$
D. $\sqrt{3}$

Exercise 21 *(MSQ) Which sets form basis of* $\mathbb{R}^3$*?*
A. $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$
B. $\{(1, 1, 1), (2, 2, 2), (3, 3, 3)\}$
C. $\{(1, 0, 1), (0, 1, 1), (1, 1, 0)\}$
D. $\{(1, 2, 3), (4, 5, 6), (7, 8, 9)\}$

1.9 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
1	CH 1.1: Vector Concepts, Properties, Dot Product and Cross Product	https://youtu.be/cMZUGkPKbso	
2	CH 1.2 - 1.3: Vector Space and Subspace	https://youtu.be/dbHC2H-Dv9Y	
3	CH 1.4 - 1.5: Col Space, Independence, Span, Basis, Rank, Null Space	https://youtu.be/ltI1tXG8yUE	
4	CH 1.6: Conceptual GATE PYQs on Systems of Linear Equations	https://youtu.be/hPLPB2v79ow	

5	CH 1.7: Solutions to Problems 1-27 — Vector Spaces	https://youtu.be/50ZK52I7NMw	

Chapter 2

Orthogonal Vectors, Subspaces, Matrices, and Projections

2.1 Orthogonal Vectors and Their Length

Concept

The length (or norm) of a vector $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ is

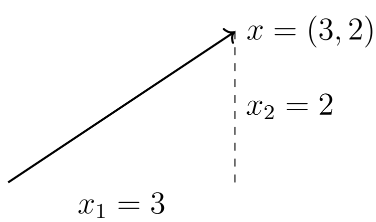
$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^T x}.$$

Example 27:

Length of $x = (1, 2, 3)$:

$$\|x\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}.$$

Example 28:



Vector length in 2D:

Concept

Two vectors $x, y \in \mathbb{R}^n$ are **orthogonal** if

$$x^T y = 0.$$

Example 29:

Check orthogonality: $x = (2, 2, -1)$ and $y = (-1, 2, 2)$:

$$x^T y = 2(-1) + 2(2) + (-1)(2) = 0$$

Hence, $x \perp y$.

Concept

A set of nonzero vectors $\{v_1, \ldots, v_k\}$ that are mutually orthogonal is automatically linearly independent.

2.2 Orthogonal Subspaces and Complements

Concept

Two subspaces $V, W \subseteq \mathbb{R}^n$ are orthogonal if every vector $v \in V$ is orthogonal to every vector $w \in W$:

$$v^T w = 0 \quad \forall v \in V, \forall w \in W.$$

Example 30:

V spanned by $v_1 = (1, 0, 0, 0), v_2 = (1, 1, 0, 0)$, W spanned by $w = (0, 0, 4, 5)$. Then $W \perp V$ because $v_i^T w = 0$.

Concept

The **orthogonal complement** of V is

$$V^\perp = \{x \in \mathbb{R}^n \mid x^T v = 0, \forall v \in V\}.$$

Concept

Fundamental Theorem of Linear Algebra (Orthogonality Version): For $A \in \mathbb{R}^{m \times n}$,

$$N(A) = (C(A^T))^\perp,$$
$$N(A^T) = (C(A))^\perp.$$

Example 31:

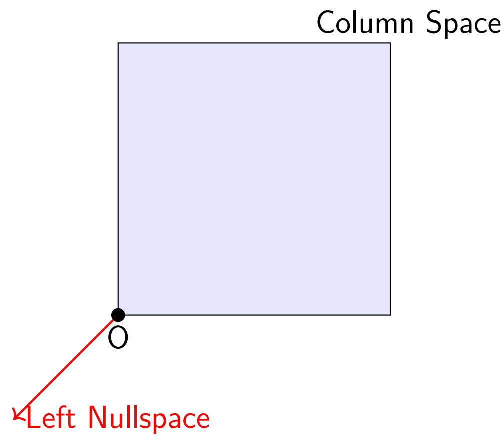
$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{bmatrix}$, rank 1. Row space multiples of $(1, 3)$, nullspace $x = (-3, 1)$.

$$Ax = 0 \Rightarrow N(A) \perp \text{Row space.}$$

Column space: line through $(1, 2, 3)$, left nullspace: plane $y_1 + 2y_2 + 3y_3 = 0$.

Example 32:

Orthogonal subspaces in 3D:



2.3 Orthogonal Matrices

Concept

$Q \in \mathbb{R}^{n\times n}$ is **orthogonal** if $Q^TQ = QQ^T = I_n$.

Concept

Properties of orthogonal matrices:

1. $Q^T = Q^{-1}$
2. $\det(Q) = \pm 1$
3. Columns (and rows) form an orthonormal set
4. $\|Qx\| = \|x\|$ for all x
5. Product of orthogonal matrices is orthogonal

Example 33:

$Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $Q^T Q = I_2$, so Q is orthogonal.

Example 34:

$\det \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} = 1.$

2.4 Projection Matrices

Concept

Projection matrix P maps $x \in \mathbb{R}^n$ onto a subspace W:

$$Px = \text{proj}_W(x)$$

If W spanned by $V = [v_1 \dots v_k]$, then

$$P = V(V^T V)^{-1} V^T$$

Concept

Properties of projection matrices:

1. $P^2 = P$ (idempotent)
2. $P^T = P$ (symmetric)
3. Eigenvalues 0 or 1
4. Rank$(P) = \dim(W)$
5. $I - P$ projects onto $W^\perp$

Example 35:

Projection onto line spanned by $v = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$:

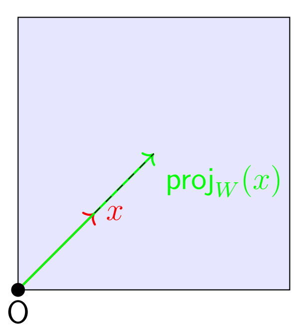
$$P = \frac{vv^T}{v^Tv} = \frac{1}{9} \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}.$$

Example 36:

Project $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto line spanned by $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$:

$$P = \frac{vv^T}{v^Tv} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad Px = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Example 37:

Projection onto a plane in 3D: 

2.5 Orthogonal and Symmetric Matrices

Concept

A square matrix $A \in \mathbb{R}^{n\times n}$ is **symmetric** if

$$A^T = A.$$

Concept

A matrix Q is both **orthogonal** and **symmetric** if

$$Q^T = Q \quad \text{and} \quad Q^T Q = I.$$

Concept

Properties of matrices that are both orthogonal and symmetric:

1. $Q^2 = I$ (since $Q^T Q = I$ and $Q^T = Q$)
2. Eigenvalues are either $+1$ or -1
3. $Q^{-1} = Q$

Example 38:

Check if

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

is orthogonal and symmetric.
Solution:

$$Q^T = Q \quad \text{(symmetric)}, \quad Q^T Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \quad \text{(orthogonal)}.$$

Hence Q is both orthogonal and symmetric. Its eigenvalues are 1 and -1.

Example 39:

If

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

then $Q^T = Q$ (symmetric) and $Q^T Q = I_2$ (orthogonal). Eigenvalues are 1 and -1.

Concept

Observation: Any orthogonal and symmetric matrix represents a **reflection** across a subspace.

2.6 Applications of Orthogonality and Projections

Concept

Linear Independence and Orthogonal Vectors: A set of nonzero vectors $\{v_1, v_2, \ldots, v_k\}$ that are mutually orthogonal is always linearly independent. This simplifies checking independence: if $v_i^T v_j = 0$ for $i \neq j$, the vectors are independent.

Example 40:

Check if the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

are linearly independent.

Solution: Compute inner products:

$$v_1^T v_2 = 0, \quad v_1^T v_3 = 1, \quad v_2^T v_3 = -1$$

Vectors are not mutually orthogonal, so we cannot immediately conclude independence. However, Gram-Schmidt can create an orthogonal basis to check independence.

Concept

Solving Systems of Linear Equations: For $Ax = b$, if columns of A are orthogonal or orthonormal, the solution is straightforward:

$$x_i = \frac{a_i^T b}{a_i^T a_i} \quad \text{for each column } a_i$$

where $A = [a_1 \ldots a_n]$.

Example 41:

Solve

$$Ax = b, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Solution: Columns of A are orthogonal.

$$x_1 = \frac{1 \cdot 3 + 0 \cdot 4}{1^2 + 0^2} = 3, \quad x_2 = \frac{0 \cdot 3 + 2 \cdot 4}{0^2 + 2^2} = 2$$

Hence, $x = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Concept

Best Approximation using Projection: If $Ax = b$ is inconsistent, the orthogonal projection of b onto the column space of A gives the vector $\hat{b}$ closest to b that is in the span of A. Then, solve $Ax = \hat{b}$ for the least-squares solution:

$$\hat{x} = (A^T A)^{-1} A^T b$$

Example 42:

Find the projection of

$$b = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

onto the column space of

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Solution:

$$\hat{x} = (A^T A)^{-1} A^T b, \quad A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad (A^T A)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 8 \\ 9 \end{bmatrix}, \quad \hat{x} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 10/3 \end{bmatrix}$$

$$\text{proj}_{\text{Col}(A)}(b) = A\hat{x} = \begin{bmatrix} 7/3 \\ 10/3 \\ 17/3 \end{bmatrix}$$

Concept

Orthogonal Decomposition: Any vector b in $\mathbb{R}^n$ can be uniquely decomposed into components along a subspace W and its orthogonal complement $W^\perp$:

$$b = \text{proj}_W(b) + (b - \text{proj}_W(b)), \quad \text{with } (b - \text{proj}_W(b)) \in W^\perp$$

Example 43:

If

$$b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad W = \text{span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\},$$

then

$$\text{proj}_W(b) = \frac{b^T v}{v^T v} v = \frac{3+4}{1+1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{7}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix}$$

$$b - \text{proj}_W(b) = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix} \in W^\perp$$

2.7 Problems

Problem 28 *(MSQ) Let* $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 2 \\ a \\ b \end{bmatrix}$ *be orthogonal. If* $\|\mathbf{v}\| = 3$*, then the possible values of* (a, b) *are:*
A. $(1, -2)$
B. $(2, -1)$

C. $(\frac{-1}{2}, \frac{-2}{3})$
D. Infinite possibilities

Problem 29 *(MCQ) Two nonzero vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *satisfy* $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\|$. *Then:*
A. $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal*
B. $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel*
C. $\|\mathbf{u}\| = \|\mathbf{v}\|$
D. Both A and C

Problem 30 *(NAT) If* $\mathbf{u}, \mathbf{v}$ *are orthogonal and* $\|\mathbf{u}\| = 2, \|\mathbf{v}\| = 3$, *find* $\|\mathbf{u}+\mathbf{v}\|$.

Problem 31 *(MSQ) Which of the following pairs of subspaces in* $\mathbb{R}^3$ *are orthogonal complements?*
A. $U = \text{span}\{(1,1,1)\},\ V = \{(x,y,z) : x+y+z=0\}$
B. $U = \text{span}\{(1,0,0)\},\ V = \text{span}\{(0,1,0),(0,0,1)\}$
C. $U = \{(x,y,z) : x-y=0\},\ V = \{(x,y,z) : x+y=0\}$
D. $U = \text{span}\{(1,1,0)\},\ V = \text{span}\{(1,-1,1)\}$

Problem 32 *(NAT) Find dim*$(U^\perp)$ *if* U *is a 2-dimensional subspace of* $\mathbb{R}^5$.

Problem 33 *(MCQ) If* Q *is an orthogonal matrix, which of the following statements is TRUE?*
A. Columns of Q *are linearly dependent*
B. $Q^TQ = I$ *but* $QQ^T \neq I$
C. Each column of Q *has unit length*
D. Determinant of Q *can be any real number*

Problem 34 *(MSQ) Let* $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. *Which of the following are true?*
A. Q *is orthogonal*
B. Q *represents a rotation by* $90°$
C. Q *is symmetric*
D. $Q^T = -Q$

Problem 35 *(MCQ) For an orthogonal matrix* Q, $\|Q\mathbf{x}\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$. *This implies that* Q*:*
A. Preserves distances but not angles
B. Preserves both distances and angles
C. Changes both distances and angles
D. Only scales the vector

Problem 36 *(NAT) Let* $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. *Compute* $\det(Q)$.

Problem 37 *(MSQ) If* P *is a projection matrix, which of the following must hold?*
A. $P^2 = I$
B. $P^2 = P$
C. $P^T = P^{-1}$
D. $\det(P) = 1$

Problem 38 *(MSQ) Which of the following matrices represent orthogonal projections in* $\mathbb{R}^2$*?*

A. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

B. $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Problem 39 *Find the projection of* $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ *onto* $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 40 *(MCQ) Let A be a symmetric matrix. Then for any $\mathbf{x}, \mathbf{y}$, $\mathbf{x}^T A\mathbf{y} = \mathbf{y}^T A\mathbf{x}$ implies:*
A. A must be orthogonal
B. A is symmetric
C. $\mathbf{x}$ and $\mathbf{y}$ are orthogonal
D. A None of these.

Problem 41 *(MSQ) Which of the following matrices are both orthogonal and symmetric?*

A. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Problem 42 *For a projection matrix P, compute the rank of $(I - P)$ in terms of rank(P) and n.*

Problem 43 *(MCQ) If A is orthogonal and symmetric, then which of the following is true?*
A. $A^2 = I$
B. $A^2 = 0$
C. $\det(A) = 0$
D. $A^{-1} \neq A$

Problem 44 *(MSQ) Let Q be an orthogonal matrix. Then which of the following are true?*
A. $\|Q\mathbf{x}\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$
B. Eigenvalues of Q have unit magnitude
C. $Q^{-1} = Q^T$
D. Q must be symmetric

Problem 45 *Find a unit vector orthogonal to both* $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Problem 46 *(MCQ) If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal nonzero vectors, then projection of $\mathbf{u}$ on $\mathbf{v}$ is:*
A. $\mathbf{u}$
B. $\mathbf{v}$
C. Zero vector
D. Depends on their magnitude

Problem 47 *(MSQ) Let $P = A(A^TA)^{-1}A^T$ be a projection matrix. Which of the following are true?*
A. $P^2 = P$
B. $P^T = P$
C. Columns of A form an orthonormal basis for Col(P)
D. P projects onto Row(A)

Problem 48 *(MCQ) If $\mathbf{v}$ is decomposed as $\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$ relative to a subspace U, then:*
A. $\mathbf{v}_{\parallel}$ *lies in* U, $\mathbf{v}_{\perp}$ *in* $U^{\perp}$
B. $\mathbf{v}_{\perp}$ *lies in* U, $\mathbf{v}_{\parallel}$ *in* $U^{\perp}$
C. Both lie in U
D. Both lie in $U^{\perp}$

Problem 49 *(NAT) Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *and* $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. *Compute the angle between them.*

Problem 50 *(MSQ) Which of the following are TRUE for orthogonal subspaces U and V in $\mathbb{R}^n$?*
A. $U \cap V = \{\mathbf{0}\}$
B. $\dim(U) + \dim(V) \leq n$
C. $(U^{\perp})^{\perp} = U$
D. $\text{Proj}_U(\text{Proj}_V(\mathbf{x})) = 0$

2.8 Try it Yourself

Exercise 22 *Check whether the vectors*

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

are linearly independent.

Exercise 23 *Determine if*

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

are orthogonal.

Exercise 24 *Find the orthogonal projection of*

$$x = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

onto the line spanned by

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Exercise 25 *Let*

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Find the least-squares solution to $Ax = b$ using projection.

Exercise 26 *Verify that the projection matrix onto*

$$v = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

is idempotent and symmetric.

Exercise 27 *Check whether the following vectors form an orthonormal set:*

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Exercise 28 *Given the matrix*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

find an orthogonal basis for its column space using Gram-Schmidt process.

Exercise 29 *Project*

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

onto the plane spanned by

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Exercise 30 *Determine if the system*

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

has a solution. If yes, find it.

Exercise 31 *If*

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

find the projection of

$$x = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

onto the subspace spanned by v_1, v_2.

2.9 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
6	CH 2.1 - 2.3: Orthogonal Vectors, Subspace, Orthogonal Matrix	https://youtu.be/x3rGhkcmVlY	
7	CH 2.4 - 2.6: Projection Vectors, Projection Matrix and Properties	https://youtu.be/22XQBWmIArY	
8	CH 2.7: Solutions to Problems 28-50 — Projection and Orthogonal	https://youtu.be/1JIW_ezUfHw	

Chapter 3

Quadratic Forms

3.1 Definition and Basic Properties

Concept

A **quadratic form** in n variables $x_1, x_2, \ldots, x_n$ is an expression of the type

$$Q(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

where $a_{ij} = a_{ji}$, i.e., $A = [a_{ij}]$ is a symmetric matrix. Equivalently, in matrix notation:

$$Q(x) = x^T A x, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad A = A^T.$$

Concept

Symmetric matrix representation: Every quadratic form can be written uniquely using a symmetric matrix:

$$Q(x) = x^T \frac{A + A^T}{2} x$$

even if A is not symmetric initially.

3.2 Diagonalization of Quadratic Forms

Concept

If A is symmetric, there exists an orthogonal matrix P such that

$$P^T AP = D$$

where D is a diagonal matrix. Then, by the change of variables $y = P^T x$, the quadratic form becomes

$$Q(x) = x^T Ax = y^T Dy = \sum_{i=1}^{n} \lambda_i y_i^2$$

where λ_i are the eigenvalues of A.

Example 44:

Diagonalize the quadratic form

$$Q(x_1, x_2) = 4x_1^2 + 4x_1x_2 + x_2^2.$$

Solution:

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, \quad \text{eigenvalues } \lambda_1 = 0, \lambda_2 = 5$$

$$\text{orthogonal matrix } P = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

Then $y = P^T x$ and $Q(x) = 0 \cdot y_1^2 + 5y_2^2 = 5y_2^2$.

3.3 Positive Definite, Negative Definite, and Indefinite Forms

Concept

Let $Q(x) = x^T Ax$.

- Positive definite: $Q(x) > 0$ for all $x \neq 0$ ($\lambda_i > 0$)
- Positive semi-definite: $Q(x) \geq 0$ ($\lambda_i \geq 0$)
- Negative definite: $Q(x) < 0$ for all $x \neq 0$ ($\lambda_i < 0$)
- Negative semi-definite: $Q(x) \leq 0$ ($\lambda_i \leq 0$)
- Indefinite: $Q(x)$ takes both positive and negative values (some $\lambda_i > 0$, some < 0)

Example 45:

Determine the definiteness of

$$Q(x_1, x_2) = 3x_1^2 + 2x_1x_2 + x_2^2$$

Solution:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{eigenvalues } \lambda_1 = 2, \lambda_2 = 2$$

All positive $\implies Q$ is positive definite.

3.4 Canonical Form Using Orthogonal Transformation

Concept

By applying an orthogonal transformation $y = P^T x$, a quadratic form can always be expressed in **canonical form**:

$$Q(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where y_i are new variables along orthogonal axes, and λ_i are eigenvalues of A.

Example 46:

Bring

$$Q(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 2x_2x_3 + x_3^2$$

to canonical form.

Solution: Compute A, find eigenvalues $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 5$, find orthogonal P, then $y = P^T x$, giving

$$Q(x) = y_2^2 + 5y_3^2$$

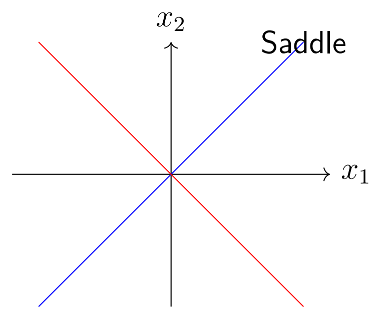
3.5 Geometric Interpretation

Concept

The canonical form aligns the axes of the quadratic form with its principal directions. The signs of λ_i determine the shape of the surface:

- Ellipsoid if all $\lambda_i > 0$
- Hyperboloid if eigenvalues have mixed signs
- Paraboloid if some $\lambda_i = 0$

Example 47:



Sketch the surface of $Q(x_1, x_2) = x_1^2 - x_2^2$.

3.6 Applications

Concept

Quadratic forms are used to:

- Test positive/negative definiteness of matrices
- Solve constrained optimization problems
- Analyze conic sections and 3D surfaces
- Diagonalize symmetric matrices (linking to orthogonal transformations)

3.7 Problems

Problem 51 (MCQ) *Consider the quadratic form*

$$Q(x_1, x_2) = 4x_1^2 + 4x_1x_2 + x_2^2.$$

Which of the following is true?

1. *Q is positive definite*
2. *Q is negative definite*
3. *Q is indefinite*
4. *Q is positive semidefinite*

Problem 52 (MSQ) *Let*

$$Q(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 + x_2^2 + x_3^2.$$

Select all that are true:

1. *Q is positive definite*
2. *Q is diagonalizable*

3. Rank$(Q) = 3$
4. Rank$(Q) = 2$

Problem 53 (NAT) *Find the rank of the quadratic form*

$$Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3$$

_____

Problem 54 (MCQ) *The quadratic form*

$$Q(x_1, x_2) = 3x_1^2 - 2x_1x_2 + x_2^2$$

is

1. *Positive definite*
2. *Negative definite*
3. *Indefinite*
4. *None of the above*

Problem 55 (MSQ) *Let*

$$Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3$$

Select all true statements:

1. Q *is positive definite*
2. *Eigenvalues are all positive*
3. Rank$(Q) = 3$
4. Rank$(Q) = 2$

Problem 56 (NAT) *Find the canonical form of*

$$Q(x_1, x_2) = 5x_1^2 + 2x_1x_2 + 2x_2^2$$

_____

Problem 57 (NAT) *Find the rank and signature of the quadratic form*

$$Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2 + 2x_1x_2$$

Rank = _____, *Signature* = _____

3.8 Try it Yourself

Exercise 32 *Check whether*

$$Q(x_1, x_2) = 2x_1^2 + 5x_2^2 - 4x_1x_2$$

is positive definite.

Exercise 33 *Diagonalize the quadratic form*

$$Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2$$

Exercise 34 *Find the canonical form of*

$$Q(x_1, x_2) = 4x_1^2 + 4x_1x_2 + x_2^2$$

Exercise 35 *Determine if*

$$Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2$$

is indefinite.

Exercise 36 *Find the rank of the quadratic form*

$$Q(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2x_1x_2$$

3.9 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
9	CH 3.1 - 3.6: Quadratic Form — Canonical — Definiteness & Indefiniteness	https://youtu.be/9za9Q-p5WPU	
10	CH 3.7: Solutions to Problems 51-57 — Quadratic Forms	https://youtu.be/vGhznjD94XA	

Chapter 4

Singular Value Decomposition (SVD)

4.1 Definition and Concept

Concept

For any real $m \times n$ matrix A, there exist orthogonal matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$, and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ such that:

$$A = U\Sigma V^T$$

where

- Columns of U are called **left singular vectors**.
- Columns of V are called **right singular vectors**.
- Diagonal entries $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$ of Σ are called **singular values** of A, where $r = \text{rank}(A)$.

4.2 Formulas and Computation

Concept

Given $A \in \mathbb{R}^{m\times n}$:

1. Compute $A^T A$ (an $n \times n$ symmetric matrix).
2. Eigenvalues of $A^T A$ are $\lambda_i = \sigma_i^2$, where σ_i are singular values.
3. Compute right singular vectors v_i from $A^T A v_i = \sigma_i^2 v_i$.

4. Compute left singular vectors $u_i = \frac{1}{\sigma_i} A v_i$.

Concept

Matrix Σ structure:

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \sigma_r & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \end{bmatrix}_{m \times n}$$

4.3 Properties of SVD

Concept

1. Rank$(A) = r =$ number of non-zero singular values.
2. $\|A\|_2 = \sigma_1$ (largest singular value).
3. $\|A\|_F^2 = \sum_{i=1}^{r} \sigma_i^2$ (Frobenius norm).
4. $AA^T = U\Sigma\Sigma^T U^T$, $A^T A = V\Sigma^T \Sigma V^T$ (diagonalization of symmetric matrices).
5. SVD always exists for any real or complex matrix.
6. Useful in low-rank approximations: $A \approx \sum_{i=1}^{k} \sigma_i u_i v_i^T$ for $k < r$.

4.4 Geometric Interpretation

Concept

SVD represents the action of A as three consecutive operations:

1. Rotate the input space using V^T.
2. Scale along coordinate axes using Σ.
3. Rotate the output space using U.

Effectively, A transforms a unit sphere into an ellipsoid whose axes lengths are the singular values σ_i and directions are columns of U.

4.5 Examples

Example 48:

Compute SVD of

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

Solution outline:

1. Compute $A^T A = \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}$.
2. Eigenvalues of $A^T A$: $\lambda = 10, 4$.
3. Singular values: $\sigma_1 = \sqrt{10}, \sigma_2 = 2$.
4. Compute v_i eigenvectors of $A^T A$.
5. Compute $u_i = \frac{1}{\sigma_i} A v_i$.
6. Construct U, Σ, V.

Example 49:

Low-rank approximation:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{rank} = 2$$

Approximate A by rank-1 matrix:

$$A_1 = \sigma_1 u_1 v_1^T$$

where $\sigma_1 = 2$, $u_1 = [1, 0, 0]^T$, $v_1 = [1, 0]^T$.

4.6 Applications

Concept

- Solving **linear systems**: $Ax = b$, including least squares for rank-deficient matrices.
- **Low-rank approximation** in data compression.
- Computing **pseudoinverse**:
$$A^+ = V\Sigma^+ U^T$$
where Σ^+ is obtained by taking reciprocal of non-zero singular values.
- **Principal Component Analysis (PCA)**.

4.7 Problems

Problem 58 *Find the Singular Value Decomposition (SVD) of*

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Determine U, Σ, and V such that $A = U\Sigma V^T$.

Problem 59 *For*

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

*find the **best rank-1 approximation** A_1 of A using its SVD.*

Problem 60 *For*

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

*find the **best rank-2 approximation** A_1 of A using its SVD.*

4.8 Try it Yourself

Exercise 37 *Compute the SVD of*

$$A = \begin{bmatrix} 4 & 0 \\ 3 & 0 \end{bmatrix}.$$

Exercise 38 *For*

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},$$

compute the singular values.

Exercise 39 *Verify that for*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad A^T A v_i = \sigma_i^2 v_i$$

holds for right singular vectors v_i*.*

4.9 YouTube Links and QR Codes

Lecture	Details	YouTube Link	QR Code
11	CH 4.1 - 4.6: Singular Value Decomposition (SVD) — Rank Approximation	https://youtu.be/gQJbrkSpFZk	

12	CH 4.7: Solutions 58-60 — Singular Value Decomposition	https://youtu.be/ZRcqKw_zf9U	

Chapter 5

Solutions

Problems Covered	YouTube Link	QR Code
Solutions to Problems 1–27 — Vectors	https://youtu.be/50ZK52I7NMw	
Conceptual GATE PYQs on Systems of Linear Equations	https://youtu.be/hPLPB2v79ow	
Solutions to Problems 28–50 — Projection and Orthogonal	https://youtu.be/1JIW_ezUfHw	

Solutions to Problems 51–57 — Quadratic Forms	https://youtu.be/vGhznjD94XA	
Solutions to Problems 58–60 — Singular Value Decomposition (SVD)	https://youtu.be/ZRcqKw_zf9U	

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